2022 Indiana Collegiate Mathematics Competition
April 9, 2022
Indiana Wesleyan University

1. Consider the four lines $l_1 : x = -3$, $l_2 : x = 1$, $l_3 : y = 2$, and $l_4 : y = -4$. If A is some point in the plane, suppose each of the segments from A to the lines meets perpendicularly at B, C, D, and E respectively (that is, B is on l_1, C on l_2, etc.). Consider the locus of all points A where

$$|AB||AC| = |AD||AE|.$$

Find an equation describing this locus, and specify as much as possible the type of plane curve it is.

Solution. This is a four-line locus problem, and so we should expect a conic section as the resulting plane curve. The required equation is

$$|x+3||1-x| = |y+4||2-y|$$

A number of teams noticed that there are actually two solutions, depending on the signs chosen for the absolute values. One is the hyperbola

$$(y+1)^2 - (x+1)^2 = 5;$$

the other is the circle

$$(x+1)^2 + (y+1)^2 = 13. \qquad \square$$

Historical note. Compare Apollonius' *Conics* III.54 and Descartes' *La Géométrie*.

2. Find two non-zero functions $f(x)$ and $g(x)$ so that $f'(x) \neq 0$, $g'(x) \neq 0$, and

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g'(x).$$

Solution. By the product rule, we need f and g to satisfy

$$f'g' = f'g + g'f.$$

For sake of simplicity, choose $f(x) = x$ so that $f'(x) = 1$. Thus we require

$$g' = g + xg'.$$

Hence

$$\frac{g'}{g} = \frac{1}{1-x}.$$

Integrating, we see that $\ln|g(x)| = -\ln|1-x| + C$, so that $|g(x)| = \dfrac{A}{1-x}$ for some constant A. Choose $A = 1$ for convenience, and suppose that g is positive. This gives us functions $f(x) = x$ and $g(x) = \dfrac{1}{1-x}$. One can easily check that these functions satisfy the requirements. Another pair of functions that work is e^{2x} and e^{2x}. $\square$

Note: This problem was inspired by 1966 Problem 4, which asks a similar question but with quotients.

3. Consider the function $f(x) = \dfrac{1}{1-e^{-1/x}}$.

   (a) (2 points) Assuming $x > 0$, find $f'(x)$.

Solution. Standard differentiation techniques yield

$$f'(x) = \frac{e^{-1/x}}{\left(1 - e^{-1/x}\right)^2 x^2}$$

□

(b) (8 points) Compute $\int_0^1 \frac{e^{-1/x}}{x^2(1-e^{-1/x})^2}\, dx$.

Solution. Observe that f is an antiderivative of the integrand, by part (a). Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_0^1 \frac{e^{-1/x}}{x^2(1-e^{-1/x})^2}\, dx &= \lim_{a\to 0^+} (f(1) - f(a)) \\ &= \frac{1}{e-1}.\end{aligned}$$

□

Note: This problem is simplification of 1971 Problem 3.

4. Let A be a square matrix, and suppose positive integers m and n exist so that $A^m = I$ and $A^n \neq I$. Find

$$\det(I + A + A^2 + \ldots + A^{m-1}).$$

Solution. (Revised Solution by Bob Foote) Let $B = I + A + A^2 + \cdots + A^{m-1}$. Observe that $B(I - A) = I - A^m = 0$. Since $A \neq I$ there is a vector $\vec{v}$ such that $A\vec{v} \neq \vec{v}$. Let $\vec{w} = (I - A)\vec{v} = \vec{v} - A\vec{v}$. Then $B\vec{w} = B(I - A)\vec{v} = 0$. Thus, $\vec{w}$ is a non-zero vector in the null space of B, and so $\det B = 0$. □

Note: This is the same as 1999 Problem 8.

5. (a) (4 points) Define $n?$ as the sum of integers from 1 to n. For example, $5? = 1 + 2 + 3 + 4 + 5$. Compute the number of zeros that appear at the end of decimal representation of 2022?.

Solution. $n?$ is a non-standard notation for the nth triangular number, given by

$$n? = T_n = \frac{(n)(n+1)}{2}.$$

Thus $2022? = 1011 * 2023$, which clearly has no zeros at the end of its decimal representation. □

(b) (6 points) Define $n!$ as the product of integers from 1 to n. For example, $5! = 1 * 2 * 3 * 4 * 5$. Compute the number of zeros that appear at the end of the decimal representation of 2022!.

Solution. Divide 2022 by powers of 5 that are less than 2022, yielding: $404 + 80 + 16 + 3 = 503$. Thus there are 503 zeros at the end of the decimal representation. □

6. Can a group be the union of two of its proper subgroups?

Solution. No. Let G be the group, and suppose by way of contradiction that H and K are proper subgroups with $G = H \cup K$. Take an element h and k from each subgroup, and consider the element hk. Then $hk \in H$ or $hk \in K$. Suppose $hk \in H$. Then $h^{-1}hk \in H$, and thus $k \in H$, a contradiction. □

Note: This is 1972 Problem 7.

7. If g is a function, denote $g \circ g \circ \ldots \circ g$ (m times) as g^m. Suppose that $g : [0, 1] \to [0, 1]$ is continuous and that there is an m so that for all x, $g^m(x) = x$. Show that in fact $g^2(x) = x$.

Solution. Note that $g(x) = g(y) \implies g^m(x) = g^m(y) \implies x = y$. Therefore g is injective. Any injective real-valued continuous function must be strictly monotone on its domain. If we suppose that g is increasing on $[0, 1]$, then for $x \in [0, 1]$,

$$x > g(x) \implies g(x) > g^2(x) \implies ... \implies g^{m-1}(x) > g^m(x),$$

hence $x > g^m(x) = x$, a contradiction. Similarly, if $x < g(x)$, we have $x < g^m(x) = x$, another contradiction. Hence $g(x) = x$. If on the other hand g is decreasing, then g^2 is increasing, and the above argument applies to show $g^2(x) = x$. Hence in either case, $g^2(x) = x$. □

Note: Compare 1977 Problem 1.

8. It is well known that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality, and the standard classroom demonstration of this involves a diagonal lines argument. Explicitly give a function between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, and show that it is bijective.

Solution. One such bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ is

$$f(m, n) = m + \frac{(m + n - 1)(m + n - 2)}{2}$$

To show bijectivity, start with letting x be any natural number. Let T_a be the largest triangular number which is smaller than x; if $x = 1$, then setting $m = n = 1$ clearly suffices. Then set

$$(m^*, n^*) = \left(x - \frac{(a)(a + 1)}{2}, a + 2 - m^*\right).$$

One can verify that $f(m^*, n^*) = x$. □

Historical note. This function is given by Cantor (1895); you can find it in *Contributions to the Founding of the Theory of Transfinite Numbers*, published by Dover.