1. Let $A = \begin{pmatrix} 1 & 2021 \\ 0 & 1 \end{pmatrix}$.

   (a) (5 points) Find A^{2021}.

   Solution: Notice that since A is upper triangular, $A^n = \begin{pmatrix} 1 & 2021 * n \\ 0 & 1 \end{pmatrix}$. Hence $A^{2021} = \begin{pmatrix} 1 & 2021^2 \\ 0 & 1 \end{pmatrix}$.

   (b) (5 points) Find a 2×2 matrix B so that $B^{2021} = A$.

   Solution: We can reverse the previous problem to see that B must also be upper triangular, and that $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ works.

2. The polynomial $p(x) = 16x^4 - 32x^3 - 104x^2 + 122x + 232$ has an interesting property. There is a line $y = mx + b$ that is tangent to the graph of p in *two* places. Find the line. Hint: consider the polynomial $q(x) = p(x) - (mx + b)$ and how it might factor.

   Solution: The polynomial $q(x)$ must factor into the form $(cx+d)^2(ex+f)^2$, because it has a double root at each point where $mx+b$ is tangent to the graph of $p(x)$. Suppose that r and s are these double roots. We use results about the roots, due to Viète. The sum of the roots, $2r+2s$, must equal $-(32/16) = 2$. Thus $s = 1 - r$. The sums of the products of the roots, $r*r+r*s+s*r+s*s = (1-r)^2 + r^2 + 4r(1-r)$ must equal $-104/16$. Solving this for r yields $r = -3/2$, so $s = 5/2$. We can then find m and b, knowing that $(r, p(r))$ and $(s, p(s))$ are the points of tangency. We get $m = 2$ and $b = 7$. Alternatively, we could use another root result: the product of the roots of q equals $(232 - b)/16$, which yields $b = 7$.

3. A three-digit positive integer n is exactly 5 times the product of its digits.

   (a) (5 points) Show that the digits of n must all be odd.

Solution: Let $P(n)$ be the product of the digits. If any digit is even, then $P(n)$ is even. Since n is also a multiple of 5, it must be a multiple of 10. Hence the one's place digit is a zero. But then $P(n) = 0$, a contradiction.

(b) (5 points) Find n.

Solution: The condition above that n must be a multiple of 5, but not a multiple of 10, means that the one's place digit must be 5. If a is the hundred's place digit and b is the ten's place digit, then $n = 100a+10b+5 = 5*5ab$. Dividing by 5, $20a+2b+1 = 5ab$. Reduce mod 5: $5ab-20a-2b-1 = 0 \mod 5$, yielding $b = 7$. We can then substitute this back in and solve for a, yielding $a = 1$. Thus $n = 175$. This is the unique solution.

4. Consider the points $A(3, 4, 1), B(5, 2, 9)$ and $C(1, 6, 5)$ in R^3. Show that these points are the vertices of a cube.

Solution: Let $AB = A - B$, etc. Then $||AB||^2 = 72$, $||AC||^2 = 24$, and $||BC||^2 = 48$. This shows that the given vertices are definitely not all on the same face of the supposed cube. If x is the edge of a cube, then the diagonal of a face is $x\sqrt{2}$ and the diagonal of the cube is $x\sqrt{3}$. The squared norms above are in a 3:2:1 ratio, which shows that AC is the edge of the cube, BC is the diagonal of a face, and AB is the diagonal of the cube.

5. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. A partition P of S into two element subsets $\{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_4, y_4\}\}$ has each x_i and each y_i a value from S, with all values used precisely once.

(a) (4 points) How many partitions of S into four two-element subsets are there?

Solution: Start with 1, and pair it any of the 7 remaining numbers. Continue with the smallest number not already used, and pair it with any of the 5 numbers remaining. Continuing, we see that the number of these partitions is $7 * 5 * 3 * 1 = 105$.

(b) (6 points) For a given partition, define its *value* by $V(P) = \sum_{i=1}^{4} x_i y_i$. We say an integer n is *achievable* if there is a partition P whose value $V(P)$ is n. Find the minimum and maximum achievable values (3 points each).

Solution: Define $N(P) = \sum_{i=1}^{4} |x_i - y_i|^2$. Then $V(P)$ is maximized whenever $N(P)$ is minimized; this occurs when the differences $x_i - y_i$ are as small as possible. The partition $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ minimizes these differences, and yields the value 100. On the other hand, maximizing $N(P)$ is achieved by making the differences as large as possible. The partition $\{\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}$ does this, yielding a value of 60.

6. Let $\underline{xyz}$ be a three-digit number, made with digits x, y, and z. That is, $\underline{xyz} = 100x + 10y + z$.

(a) (5 points). Find digits a, b, and c, not necessarily distinct, for which $\underline{abc} + \underline{cab} - \underline{bca} = \underline{608}$.

Solution: Let $a = 3$, $b = 4$, and $c = 7$.

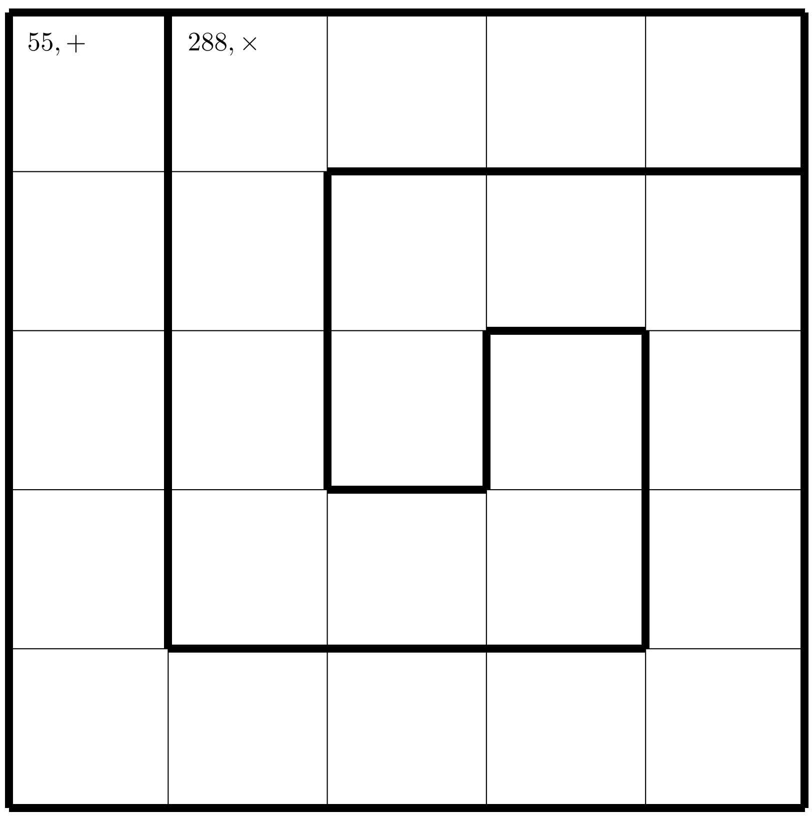
(b) (5 points). Show that there are no values of a, b, and c that satisfy $\underline{abc} + \underline{cab} - \underline{bca} = \underline{707}$.

Solution: The left-hand side is equivalent to $109a + 91c - 89b$, which must equal 707 by the conditions of the problem. Rewriting, $2(10a+b) = 707 - 89(a+c-b)$. The left-hand side is even, which means $n = a + c - b$ must be odd. It is easy to see that $n < 8$, since $707 - 89 * 8 < 0$. Since a and b are single digits, the largest that $(10a+b)$ can be is 99, which means only $n = 7$ is a possibility. This makes $(10a + b) = 42$, making $a = 4$ and $b = 2$, and hence $c = 5$. However, this makes $109a + 91c - 89b = 713$, not 707 as required.

7. Polynomial p has non-negative integer coefficients, and satisfies $p(1) = 21$ and $p(11) = 2021$. What is $p(10)$?

Solution: Let $p(x) = \sum_{k=0}^{n} a_k x^k$. The condition that $p(1) = 21$ means the coefficients must sum to 21, and hence each of the coefficients cannot exceed 21. Assume for the moment that each coefficient is a single digit. The condition that $p(11) = 2021$ means that 2021 is the base-10 value of a number with most significant digit a_n and one's digit a_0, expressed in base-11. The question of what $p(10)$ is, means that we are seeking a number expressed in base-10 whose most significant digit is a_n and whose one's digit is a_0. This is equivalent to asking what the base-11 representation of 2021 is, which is 1578. We can then verify that $1 + 5 + 7 + 8 = 21$.

8. In a "KenKen" puzzle, the numbers in each heavily outlined set of squares, called cages, must combine (in any order) to produce the target number in the top corner of the cage using the mathematical operation indicated. A number can be repeated within a cage as long as it is not in the same row or column. In the 5×5 puzzle, each of the digits 1 through 5 must appear precisely once in each row and column. Solve the 5×5 KenKen below.



Solution: As a preliminary to solving the puzzle, you may find it helpful to see that either completely filling in either cage determines the solution to the other cage.

a	b	d	c	e
d	e	b	a	c
c	d	a	e	b
b	c	e	d	a
e	a	c	b	d

The number 288 has a unique factorization in exactly 10 numbers from the set $\{1, 2, 3, 4, 5\}$. $288 = 1^4 2^3 3^2 4^1 5^0$. Next, see the letters above and note that $e = 1, d = 2, c = 3, b = 4, a = 5$ is the only way to build the number 288. Then verify that the other cage's clue is also satisfied.