Spring 2019 Indiana Collegiate Mathematics Competition (ICMC) Solutions

Mathematical Association of America – Indiana Section

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Provide counterexamples to each of the following statements:

(1) If both $f(x)$ and $g(x)$ are continuous and monotone on $\mathbb{R}$, then $f(x)+g(x)$ is continuous and monotone on $\mathbb{R}$.

Solution: Let $f(x) = x + \sin(x)$ and $g(x) = -x$. Then f and g are both continuous. Individually, both functions are monotone, but their sum $(f+g)(x) = \sin(x)$ is not.

(2) If a function $f(x)$ is not bounded in any neighborhood of a point a, then either $\lim_{x\to a^+} |f(x)| = \infty$ or $\lim_{x\to a^-} |f(x)| = \infty$.

Solution: The function $f(x) = \dfrac{\sin(\frac{1}{x})}{x}$ is not bounded in any neighborhood of $x = 0$, and neither of the one-sided limits at $x = 0$ exist.

(3) A function cannot be continuous at only one point in its domain and discontinuous everywhere else.

Solution: The function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

is continuous at $x = 0$, but discontinuous everywhere else.

(4) If a function is differentiable and increasing on an interval (a, b), then its derivative is positive on the interval (a, b).

Solution: The function $f(x) = x^3$ has $f'(0) = 0$, yet the function is increasing on $\mathbb{R}$.

(5) If $f(x)$ is a function with an antiderivative $F(x)$ that is defined at both a and b, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Solution: If $f(x) = \frac{1}{x}$, then its antiderivative $F(x) = \ln|x|$ is defined at both $x = -1$ and $x = 1$, but $\int_{-1}^{1} \frac{1}{x}\,dx$ does not converge.

Let $f(x) = \frac{x^2}{4}$, and consider the set of all right triangles in the plane whose right angle vertex lies at the origin and whose other two vertices lie somewhere else on the graph of $y = f(x)$. Conjecture a point other than the origin that all such triangles must pass through, and then prove your conjecture.

Solution: All such triangles must pass through the point $(0, 4)$. To see this, first note that since the right angle occurs at the origin, the legs of the right triangle are formed by perpendicular lines whose equations can be written as $y = mx$ and $y = \dfrac{-x}{m}$ for some $m > 0$. These lines intersect the parabola at the points $(4m, 4m^2)$ and $(\dfrac{-4}{m}, \dfrac{4}{m^2})$ respectively. Thus, the equation for the hypotenuse of the triangle is

$$y = \left(\frac{4m^2 - \frac{4}{m^2}}{4m + \frac{4}{m}}\right)(x - 4m) + 4m^2$$

$$y = \left(\frac{4m^2 - \frac{4}{m^2}}{4m + \frac{4}{m}}\right)x + 4$$

No matter what m is, when x equals 0, y will equal 4. Thus, the hypotenuse of the right triangle will always pass through the point $(0, 4)$.

Two-Face and his henchmen are once again on a crime spree terrorizing the good people of Gotham City. However, rather than flipping a coin to decide his behavior, Two-Face has decided to adopt a more deterministic approach. When Two-Face's gang crashes a party with 60 guests in attendance, they first line everyone up against the wall. Then, the first henchman walks down the line and takes \$10 from every guest. Next, a second henchman walks down the line giving \$10 to the second, fourth, sixth, etc. party guests. This process is repeated with a third henchman taking \$10 from the third guest and every third guest thereafter, then with a fourth henchman giving \$10 to the fourth guest and every fourth guest thereafter, and so on, until finally the sixtieth henchman gives \$10 to only the sixtieth (last) person in line.

(1) How many party guests made money? How many lost money? How many broke even?

Solution: Note that the amount of money that party guest n makes (or loses) is given by $f(n) = 10(e_n - o_n)$, where e_n denotes the number of even positive divisors of n and o_n denotes the number of odd positive divisors of n. Thus, if n is odd, guest n will lose money since $e_n = 0$. If n is even but not divisible by 4, guest n will break even, since every even divisor of n can be paired with an odd divisor of n. Finally, if n is divisible by 4, party guest n will make money since n will have more even divisors than odd divisors since every odd divisor is paired with an even divisor but not every even divisor is paired with an odd divisor. Thus, 30 guests will lose money, 15 will break even, and 15 will make money.

(2) Which party guest(s) made the most money?

Solution: By the previous part, we only need to consider guests whose number n is divisible by 4. The party guest who makes the most money will correspond to the number n for which the number of even divisors exceeds the number of odd divisors by the most. Of the numbers divisible by 4, the best candidates are $2^5 = 32$, $2^4 \cdot 3 = 48$, $2^3 \cdot 5 = 40$, $2^3 \cdot 3 = 24$, and $2^2 \cdot 3 \cdot 5 = 60$. Of these, we see that $f(24) = f(32) = f(40) = f(60) = 40$, while $f(48) = 60$. Thus, the 48th party guest in line will make the most money.

(3) Which party guest(s) lost the most money?

Solution: By the first part, we only need to consider guests whose number n is odd. The party guest who loses the most money will correspond to the number n which has the most odd divisors. The best candidates are $3^2 \cdot 5 = 45$ and $3^3 = 27$. Since $f(45) = -60$ and $f(27) = -40$, the guest who lost the most would be the 45th guest in line.

(4) How much money did Two-Face and his henchmen make?

Solution: Since $\sum_{n=1}^{60} f(n) = -390$, Two-Face and his henchmen made \$390. Note that by the first part, you only need to consider those values of n that are odd or that are divisible by 4 when computing this sum.

A fair coin is a coin that will produce a result of either heads (H) or tails (T) when flipped with equal probability.

(1) If you start flipping a fair coin, what is the expected number of flips needed to get your first tail (T)?

Solution: Let X denote the number of flips needs to get your first tail. Flipping a H on your first toss adds one to your required number of tosses and gets you no closer to your goal. Meanwhile, flipping a T on your first toss completes your task in a single toss. Thus,

$$E(X) = P(H)E(X|H) + P(T)E(X|T)$$
$$E(X) = P(H)(E(X) + 1) + P(T)(1)$$
$$E(X) = \frac{1}{2}(E(X) + 1) + \frac{1}{2}$$
$$\frac{1}{2}E(X) = 1$$
$$E(X) = 2$$

Alternatively,

$$E(X) = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 2.$$

(2) If you start flipping a fair coin, what is the expected number of flips needed to achieve your first string of heads followed by tails (HT)?

Solution: Let Y denote the number of flips needed to achieve your first string of heads followed by tails. Flipping a T on your first toss adds one to your required number of tosses and gets you no closer to your goal. If you flip a H on your first toss, you are just waiting on your first T. Let X denote the number of additional tosses required to get your first T if your first toss is H. Similarly to the previous part, we can see that $E(X) = 2$. Then

$$E(Y) = P(T)E(Y|T) + P(H)E(Y|H)$$
$$E(Y) = P(T)(E(Y) + 1) + P(H)(E(X) + 1)$$
$$E(Y) = \frac{1}{2}(E(Y) + 1) + \frac{1}{2}(2 + 1).$$
$$\frac{1}{2}E(Y) = 2$$
$$E(Y) = 4.$$

Alternatively,

$$E(Y) = \sum_{n=1}^{\infty} n \left(\frac{n-1}{2^n}\right) = 4.$$

(3) If you start flipping a fair coin, what is the expected number of flips needed to achieve the first string of heads followed by heads (HH)?

Solution: Let Z denote the number of flips needed to achieve the first string of heads followed by heads. Flipping a T on your first toss adds one to your number of required flips but does not get you closer to getting two heads. Likewise, starting out by flipping HT adds two flips to your total but does not get you closer to getting two heads. Meanwhile, starting out with HH takes two flips and completes the task in two tosses. Thus,

$$
\begin{aligned}
E(Z) &= P(T)E(Z|T) + P(HT)E(Z|HT) + P(HH)E(Z|HH) \\
E(Z) &= P(T)(E(Z)+1) + P(HT) * (E(Z)+2) + P(HH)(2) \\
E(Z) &= \frac{1}{2}(E(Z)+1) + \frac{1}{4}(E(Z)+2) + \frac{1}{4}(2) \\
\frac{1}{4}E(Z) &= \frac{3}{2} \\
E(Z) &= 6.
\end{aligned}
$$

Let $A = \begin{pmatrix} 1 & 2019 \\ 0 & 1 \end{pmatrix}$. For **any** $n \in \mathbb{Z}$, conjecture what the four entries of the matrix A^n are, and then prove your conjecture.

Solution:

Claim: For any integer n, $A^n = \begin{pmatrix} 1 & 2019n \\ 0 & 1 \end{pmatrix}$.
We will prove this using double induction. For the base cases, first observe that $A^1 = \begin{pmatrix} 1 & 2019 * 1 \\ 0 & 1 \end{pmatrix}$, $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and that $A^{-1} = \begin{pmatrix} 1 & -2019 \\ 0 & 1 \end{pmatrix}$ since

$$\begin{pmatrix} 1 & 2019 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2019 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Next, assume that for some positive integer k, $A^k = \begin{pmatrix} 1 & 2019k \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \begin{pmatrix} 1 & 2019k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2019 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 * 1 + 2019k * 0 & 2019 + 2019k \\ 0 * 1 + 1 * 0 & 0 * 2019 + 1 * 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2019(k+1) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Finally, assume that for some negative integer k, $A^k = \begin{pmatrix} 1 & 2019k \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} A^{k-1} &= A^k A^{-1} \\ &= \begin{pmatrix} 1 & 2019k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2019 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 * 1 + 2019k * 0 & -2019 + 2019k \\ 0 * 1 + 1 * 0 & 0 * -2019 + 1 * 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2019(k-1) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This completes our proof.

Let $(s_n)_{n=1}^{\infty}$, $(t_n)_{n=1}^{\infty}$, and $(u_n)_{n=1}^{\infty}$ be sequences with the following properties:

i) $(s_n)_{n=1}^{\infty}$ is monotone decreasing
ii) $(t_n)_{n=1}^{\infty}$ is monotone increasing
iii) $s_n \geq u_n \geq t_n$ for every $n \in \mathbb{N}$.

For each of the three sequences $(s_n)_{n=1}^{\infty}$, $(t_n)_{n=1}^{\infty}$, and $(u_n)_{n=1}^{\infty}$, conjecture whether that sequence must converge, must not converge, or if not enough information is given to determine convergence. Then prove your conjectures.

Solution: The sequences $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ must converge, while not enough information is provided to determine if the sequence $(u_n)_{n=1}^{\infty}$ converges.

To see that $(s_n)_{n=1}^{\infty}$ must converge, note that it is a monotone decreasing sequence that is bounded below by t_1 since for all $n \in \mathbb{N}$, $s_n \geq t_n$ by iii) and $t_n \geq t_1$ by ii). Every monotone decreasing sequence that is bounded below converges.

To see that $(t_n)_{n=1}^{\infty}$ must converge, note that it is a monotone increasing sequence that is bounded above by s_1 since for all $n \in \mathbb{N}$, $t_n \leq s_n$ by iii) and $s_n \leq s_1$ by i). Every monotone increasing sequence that is bounded above converges.

Not enough information is given to determine if the sequence $(u_n)_{n=1}^{\infty}$ will converge. There are some examples that satisfy all the given criteria for which $(u_n)_{n=1}^{\infty}$ will converge, and others for which it will not converge. For example, if $s_n = 1 + \frac{1}{n}$, $t_n = -1 - \frac{1}{n}$, and $u_n = 0$, then $(s_n)_{n=1}^{\infty}$ is a monotone decreasing sequence, $(t_n)_{n=1}^{\infty}$ is a monotone increasing sequence, and $s_n \geq u_n \geq t_n$ for all $n \in \mathbb{N}$, and $(u_n)_{n=1}^{\infty}$ does converge. Meanwhile, if $s_n = 1 + \frac{1}{n}$, $t_n = -1 - \frac{1}{n}$, and $u_n = (-1)^n$, then $(s_n)_{n=1}^{\infty}$ is a monotone decreasing sequence, $(t_n)_{n=1}^{\infty}$ is a monotone increasing sequence, and $s_n \geq u_n \geq t_n$ for all $n \in \mathbb{N}$, but $(u_n)_{n=1}^{\infty}$ does not converge.

Suppose that G is a group. A subset S of G is called a set of *generators* for G if every element of G can be written as the finite product of elements in S and/or their inverses.

Suppose that G is a group with identity e which has $\{x, y\}$ as a set of generators, where the generators x and y satisfy the following relations:

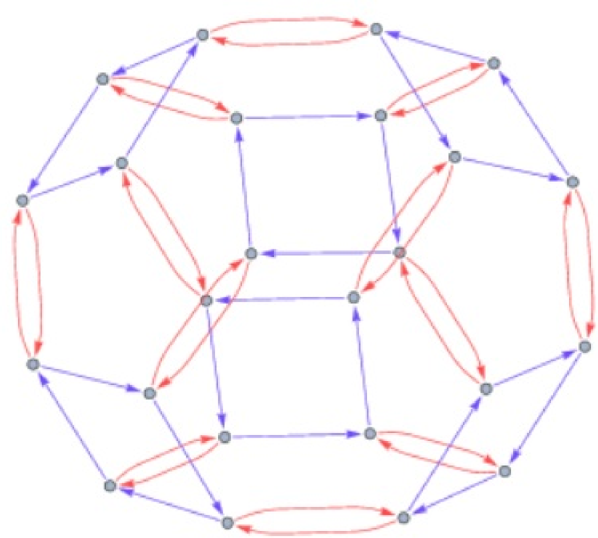
i) $x^2 = e$
ii) $y^4 = e$
iii) $xyxyxy = e$.

Determine the maximum number of elements that G can contain. Justify your answer.

Solution: Since x and y generate G, every element of G can be written as a finite product of x's, y's, x^{-1}'s, and y^{-1}'s. Based on relations i) and ii) given above, $x^{-1} = x$ and $y^{-1} = y^3$, so every element of G can be written as a finite product of just x's and y's, where you never need more than one x in a row or more than three y's in a row. Based on this information and relation iii), every element in G must be equivalent to at least one of the following 24 products:

$$e, y, y^2, y^3, x, xy, xy^2, xy^3, yx, yxy, yxy^2, yxy^3, y^2x, y^2xy,$$
$$y^2xy^2, y^2xy^3, y^3x, y^3xy, y^3xy^2, y^3xy^3, xy^2x, xy^2xy, xy^2xy^2, xy^2xy^3.$$

One way to visualize this is via the following graph (called a Cayley graph) that illustrates the relations between the generators x and y.



Thus, the maximum number of elements that G can contain is 24.

Suppose that T is the triangular pyramid with vertices at $(0,0,0)$, $(12,0,0)$, $(0,8,0)$, and $(0,0,24)$. What is the maximum volume that a rectangular prism R which has one vertex at $(0,0,0)$ and which is inscribed in T can have?

Solution: The faces of the pyramid T are determined by the xy-plane, the xz-plane, the yz-plane, and the plane P defined by $z = -2x - 3y + 24$. Since one corner of the rectangular prism R must be at the origin, the volume of R will be maximized when the opposite corner is on the plane P. If this corner is located at the point $(x, y, -2x - 3y + 24)$ on the plane P, then the volume of R is given by

$$V(x,y) = xy(-2x - 3y + 24) = -2x^2y - 3xy^2 + 24xy.$$

Taking partial derivatives, we get that

$$V_x(x,y) = -4xy - 3y^2 + 24y = y(-4x - 3y + 24)$$

and that

$$V_y(x,y) = -2x^2 - 6xy + 24x = x(-2x - 6y + 24).$$

Setting these partial derivatives equal to zero and solving for x and y, we see that $(x,y) = (4, \frac{8}{3})$ is one critical point of V. Using the second partial derivative test, we can see that this critical point corresponds to a maximum for V. This means that the maximum volume would be $V = 4(\frac{8}{3})(-2(4) - 3(\frac{8}{3}) + 24) = 4(\frac{8}{3})(8) = 85\frac{1}{3}$ cubic units.