Spring 2018 Intersectional Collegiate Mathematics Competition (ICMC) Exam

Mathematical Association of America – Illinois, Indiana, and Michigan Sections

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Show that

$$\sin(x)\sin(2x)...\sin(nx) \neq 1$$

for every real number x and any positive integer $n \geq 2$.

Solution:
For the above equality to happen one needs $|\sin(nx)| = 1$ for any n. Since $n \geq 2$ we have

$$|\sin(x)| = 1 \text{ and } |\sin(2x)| = 1$$

which leads to

$$x = \frac{\pi}{2} + k\pi \text{ and } 2x = \frac{\pi}{2} + l\pi \text{ with } k,\ l \text{ integers}$$

Combining the two one obtains

$$l - 2k = \frac{1}{2}$$

which is a contradiction.

Determine the smallest natural number n such that

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}} \geq 100.$$

Solution:
Rationalizing the denominators one obtains the following telescopic summation:

$$\sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \cdots + \sqrt{n+1} - \sqrt{n} \geq 100$$

which simplifies to

$$\sqrt{n+1} - 1 \geq 100 \text{ and } n \geq 10200.$$

Hence the smallest n is 10200.

Suppose a_1, a_2,...,a_n are strictly positive real numbers and
$a_1^x + a_2^x + ... + a_n^x \geq n$ for every real number x. Prove that $a_1 a_2 \cdots a_n = 1$.

Solution:
Consider the function $f(x) = a_1^x + a_2^x + ... + a_n^x$. Notice that $f(0) = n$ and, from the hypothesis, $f(x) \geq f(0) = n$. Therefore $x = 0$ is a local minimum. From Fermat's theorem it follows that $f'(0) = 0$. But

$$f'(x) = a_1^x \ln(a_1) + ... + a_n^x \ln(a_n).$$

Then

$$f'(0) = \ln(a_1) + ... + \ln(a_n) = \ln(a_1 a_2 \cdots a_n) = 0.$$

and

$$a_1 a_2 \cdots a_n = 1.$$

Show that if $x + y + z > 0$ then

$$\det \begin{bmatrix} x & z & y \\ y & x & z \\ z & y & x \end{bmatrix} \geq 0.$$

Solution:
First we use the following row and column operations which will not change the value of the determinant: Add rows 2 and 3 to row 1 and, then, subtract column 1 from column 2 and column 1 from column 3. With these operations the determinant becomes

$$\begin{vmatrix} x+y+z & 0 & 0 \\ y & x-y & z-y \\ z & y-z & x-z \end{vmatrix} = (x+y+z) \begin{vmatrix} 1 & 0 & 0 \\ y & x-y & z-y \\ z & y-z & x-z \end{vmatrix}$$

which is

$$(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

and, further,

$$\frac{1}{2}(x+y+z)[(x-y)^2+(y-z)^2+(z-x)^2] \geq 0.$$

Consider the following sequence defined recursively

$$x_1 = \frac{1}{2}, \quad x_{k+1} = x_k^2 + x_k, \quad k \geq 1.$$

Find the integer part of S_{100} where

$$S_{100} = \frac{1}{x_1+1} + \frac{1}{x_2+1} + ... + \frac{1}{x_{100}+1}.$$

Solution:
Notice

$$\frac{1}{x_{k+1}} = \frac{1}{x_k} - \frac{1}{x_k+1} \text{ or } \frac{1}{x_k+1} = \frac{1}{x_k} - \frac{1}{x_{k+1}}.$$

Adding up for $k = 1..100$ we get a telescoping cancellation and

$$S_{100} = \frac{1}{x_1} - \frac{1}{x_{101}} = 2 - \frac{1}{x_{101}}.$$

Since x_n is increasing and $x_3 > 1$ we have $0 < \frac{1}{x_{101}} < 1$ and $S_{100} = 2 - \frac{1}{x_{101}} < 2$.
So $\lfloor S_{100} \rfloor = 1$.

Consider a semicircle and AB its diameter. Pick two arbitrary points D and E on the semicircle such that the segments (AD) and (BE) intersect at M in the interior of the semicircle. Prove that

$$|AM| \cdot |AD| + |BM| \cdot |BE| = |AB|^2.$$

Solution:
First notice that,by dividing by $|AB|^2$, the equality to prove becomes

$$\frac{|AM|}{|AB|}\frac{|AD|}{|AB|} + \frac{|BM|}{|AB|}\frac{|BE|}{|AB|} = 1.$$

Consider the perpendicular from M to AB and denote by P the intersection of this perpendicular with AB.

Notice that the triangles AMP and ABD are similar and the ratio of the segments opposite the congruent angles is the same. Hence

$$\frac{|AM|}{|AB|} = \frac{|AP|}{|AD|}.$$

Analogously, notice that the triangles BMP and BAE are also similar and, from there, we infer that

$$\frac{|BM|}{|AB|} = \frac{|PB|}{|BE|}.$$

Substituting these into the equality to prove we obtain

$$\frac{|AP|}{|AD|}\frac{|AD|}{|AB|} + \frac{|PB|}{|BE|}\frac{|BE|}{|AB|} = \frac{|AP| + |PB|}{|AB|} = \frac{|AB|}{|AB|} = 1.$$

Suppose a fair six-sided die is rolled and let X denote the outcome of the die roll. Suppose a fair coin is flipped until X heads are obtained. Compute the expected value of the number of flips.

Solution:
Let $Y =$ the number of coin flips. On average, it takes 2 flips to obtain a head. Hence, on average, it takes $2X$ flips to obtain X heads.

$$E[Y] = E[E[Y|X]] = E[2X] = 2E[X] = 2(3.5) = 7.$$

A town has n inhabitants who like to form clubs. They want to form clubs so that every pair of clubs should share a member, but no three clubs should share a member. What is the maximum number of clubs they can form? Illustrate with an example.

Solution:
Let c be the number of clubs. Since each pair of clubs (but not three) share a member, the number of pairs of clubs cannot be more than the number of people. In other words, $\binom{c}{2} \leq n$. This condition becomes

$$c^2 - c - 2n \leq 0$$

which is a quadratic in c concave upward with two real roots, one positive and one negative. Therefore c must be between zero and the positive root, i.e. $c \leq (1 + \sqrt{1 + 8n})/2$.

To achieve the equality with the integer part of the bound assign a unique person to each pair of clubs. Each club will have its members based on these assignments. For example, if $n = 6$, then $c = 4$. Let the people be denoted by A, B, C, D, E, F, the clubs by $1, 2, 3, 4$. Based on the assignment

A in $1, 2$ B in $1, 3$ C in $1, 4$ D in $2, 3$ E in $2, 3$ F in $3, 4$,

the members of the clubs are as follows:

1: A, B, C 2: A, C, E 3: B, C, F 4: D, E, F.