Spring 2017
Indiana Collegiate Mathematics Competition (ICMC)
Solutions

Mathematical Association of America – Indiana Section

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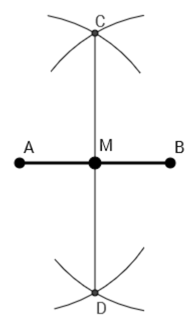
In constructible geometry, one constructs points, lines, and circles from given points, lines, and circles, using an unmarked straight edge and compass.

- The straight edge draws a line between points already given, which includes the line segment connecting them; the line may extend as far beyond either point as desired. New points are created where the line intersects already existing lines or circles.
- The compass draws a circle (or a circular arc) centered on a given point with a radius extending to another point from the center. Again, new points are created where the circle intersects other circles or lines.

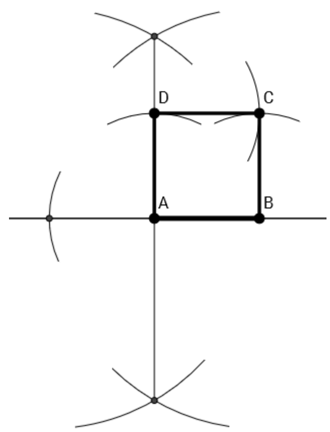
(a) Use the provided straight edge and compass to construct the midpoint M of the line segment $\overline{AB}$ given below.

Solution: Draw a circle centered at A that passes through B, as well as a circle centered at B that passes through A. These two circles will intersect in two points which we'll call C and D. Then draw the line segment connecting C and D. This line segment will intersect $\overline{AB}$ in a point that we'll call M. M is the midpoint of $\overline{AB}$. See the figure below.

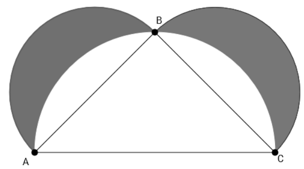


(b) Use the provided straight edge and compass to construct a square $ABCD$ with the line segment $\overline{AB}$ given below as one of its sides. Do not erase any intermediate steps in your construction.

Solution: Use the straight edge to extend $\overline{AB}$ to the left. Then draw the circle centered at A that passes through B. This circle will intersect $\overleftrightarrow{AB}$ at B and another point. Draw a circle centered at this new point that passes through B, as well as a circle centered at B that passes through this new point. Then draw the line segment that connects the two points where these circles intersect. This line segment passes through A and is perpendicular to $\overline{AB}$. The corner D of our square will be one of the points where this line segment intersects the circle centered at A that passes through B. Finally, draw a circle centered at D that passes through A and a circle centered at B that passes through A. These two circles will intersect at A and another point, namely the final corner C of our square. To complete the square, draw the line segments connecting C with D and C with B. See the figure on the top of the next page.

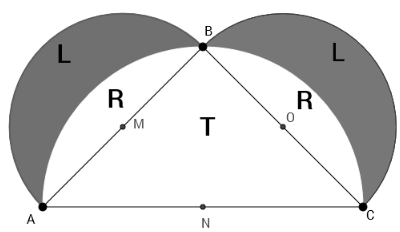


(c) Given an isosceles right triangle ABC, one can use a compass to construct lunes as follows. First, one semicircle is formed with the line segment $\overline{AC}$ as its diameter. Two other semicircles are formed with $\overline{AB}$ and $\overline{BC}$ as their diameters. The lunes are the shaded shapes in the figure below.



Use the provided straight edge and compass to construct a square whose area is equal to the area of **one** of the lunes on the diagram given below. Justify how you know the square you construct has the appropriate area.

Solution: Let s denote the length of $\overline{AB}$. Since triangle ABC is a right isosceles triangle, the length of $\overline{AC}$ is then $\sqrt{2}s$.



If T, R, and L denote the areas of the indicated regions in the above figure, $T = \frac{1}{2}s^2$ since it is the area of a triangle with base s and height s,

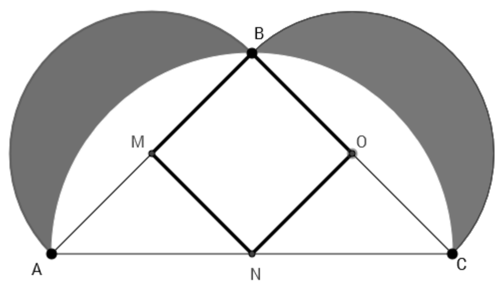
$$T + 2R = \frac{1}{2}\pi\left(\frac{\sqrt{2}s}{2}\right)^2 = \frac{\pi}{4}s^2$$

since $T + 2R$ is the area inside a semicircle with radius $\frac{\sqrt{2}s}{2}$, and

$$L + R = \frac{1}{2}\pi\left(\frac{s}{2}\right)^2 = \frac{\pi}{8}s^2$$

since $L+R$ is the area inside a semicircle with radius $\frac{s}{2}$. We can use the first two equations to see that $R = \frac{\pi}{8}s^2 - \frac{1}{4}s^2$. We can then use this value for R as well as the third equation given above to see that L, which is the area of the lune, is

$L = \frac{1}{4}s^2$. Thus, the area of the lune will equal the area of a square that has side length $\frac{1}{2}s$, which is just the length of the line segment from B to the midpoint of $\overline{AB}$. Therefore, to construct a square with area equal to the area of one of the lunes, we can use our construction from part (a) to construct the midpoint M of $\overline{AB}$ and then use our construction from part (b) to construct a square that has $\overline{MB}$ as one of its sides. (Alternatively, since triangle ABC is a right isosceles triangle, the polygon that connects B, the midpoint M of $\overline{AB}$, the midpoint N of $\overline{AC}$, and the midpoint O of $\overline{BC}$ will be a square with side length $\frac{1}{2}s$.)



Consider the following series
$$\sum_{n=1}^{\infty}(a_n)^n$$
where
$$a_n = \begin{cases} \frac{1}{3^1} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} & \text{if } n \text{ is odd.} \\ |\sin n \cos n| & \text{if } n \text{ is even.} \end{cases}$$

Make a conjecture as to whether the series converges or not, and then prove your conjecture.

Solution:

This series does converge. To see this, we make a comparison to the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$, which is a convergent geometric series, since its ratio is $\frac{1}{2} < 1$. We demonstrate that $0 \leq (a_n)^n \leq \left(\frac{1}{2}\right)^n$ for all n.

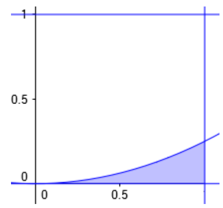
If n is odd, then a_n is the n^{th} partial sum of the geometric series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$, which converges to $\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$. Since the terms of series are non-negative, it follows that $0 \leq a_n \leq \frac{1}{2}$. Thus $0 \leq (a_n)^n \leq \left(\frac{1}{2}\right)^n$, as we desire for the comparison test.

If n is even, then $a_n = |\sin n \cos n| = \frac{1}{2}|\sin 2n|$. Since $|\sin 2n| \leq 1$ for all n, $0 \leq a_n \leq \frac{1}{2}$. Then $0 \leq (a_n)^n \leq \left(\frac{1}{2}\right)^n$ for all n, as we desire for the comparison test.

We've shown that for all n (odd or even), $0 \leq (a_n)^n \leq \left(\frac{1}{2}\right)^n$. Hence, by the comparison test, $\displaystyle\sum_{n=1}^{\infty}(a_n)^n$ converges.

Two students, Joe and Frank, are each asked to independently select a number at random from the interval [0,1] in such a way that each number in [0,1] is just as likely to be chosen as any other number in this interval. If a denotes the number chosen by Joe and b denotes the number chosen by Frank, what is the probability that the quadratic equation $x^2 + ax + b = 0$ has at least one real root?

Solution:
Note that $x^2+ax+b=0$ has at least one real root as long as the discriminant, which is $a^2 - 4b$ in this case, is non-negative. Thus, we need to find the probability that $a^2-4b \geq 0$, i.e. $b \leq \frac{1}{4}a^2$. Since each student is picking a number from $[0,1]$ according to the uniform probability distribution, finding the desired probability is equivalent to finding the proportion of $[0,1] \times [0,1]$ where $b \leq \frac{1}{4}a^2$. This region is shaded in the figure below, where the horizontal axis corresponds to a and the vertical axis corresponds to b.



The area of the shaded region is

$$\int_0^1 \frac{1}{4}a^2 \, da = \frac{1}{12},$$

while the area of the entire square is 1. Thus, the desired probability is the ratio of these areas, namely $\frac{1}{12}$.

A *multiplicative magic square* is an $n \times n$ square array of numbers consisting of n^2 distinct positive integers (not necessarily consecutive) arranged such that the product of the n numbers in any of the n rows, n columns, or 2 main diagonal lines is always the same number. Call this common product the *magic product*.

(a) Show that the magic product of a 3×3 multiplicative magic square must be a perfect cube.

Solution:
Suppose that the diagram below represents a 3×3 multiplicative magic square with magic product P.

a	b	c
d	e	f
g	h	i

Then $P = abc = def = ghi = adg = beh = cfi = aei = gec$. Note that $a = \frac{P}{ei}$, $b = \frac{P}{eh}$, and $c = \frac{P}{eg}$. Thus,

$$\begin{aligned} P &= abc \\ &= \left(\frac{P}{ei}\right)\left(\frac{P}{eh}\right)\left(\frac{P}{eg}\right) \\ &= \frac{P^3}{e^3 ghi} \\ &= \frac{P^3}{e^3 P} \\ &= \frac{P^2}{e^3}. \end{aligned}$$

Solving $P = \frac{P^2}{e^3}$ for P, we see that $P = e^3$, making P a perfect cube.

(b) Find an example of a 3×3 multiplicative magic square whose magic product is minimal. Explain how you know this magic product is minimal.

Solution: By the previous part, we know that the magic product must be a perfect cube. Moreover, the magic product must have at least 9 distinct divisors because each entry in the magic square is a divisor of the magic product. The first five perfect cubes, 1, 8, 27, 64, and 125, have 1, 4, 4, 7, and 4 divisors, respectively. The next perfect cube, 216, has 16 divisors, so it is a candidate for the minimal magic product. Since the sample array given below is a 3×3 multiplicative magic square with magic product 216, 216 is, in fact, the minimal magic product.

12	1	18
9	6	4
2	36	3

Other multiplicative magic squares with magic product 216 are possible. However, our solution to the previous part implies that all of them will have a 6 in the middle square.

For any positive integer n, let $s(n)$ be the sum of the first n terms of the sequence

$$0, 1, 1, 2, 2, 3, 3, 4, 4, \ldots, k, k, k+1, k+1, \ldots$$

(a) Find a formula for $s(n)$. (Note: Your final formula should not have "$\ldots$" in it.)

Solution:
This formula relies on the fact that for any positive integer k, the sum of the integers from 1 to k is $\frac{1}{2}k(k+1)$.

We'll first consider the case when n is odd. In this situation,

$$\begin{aligned}
s(n) &= 0 + 1 + 1 + 2 + 2 + \cdots + \frac{n-1}{2} + \frac{n-1}{2} \\
&= 2\left(1 + 2 + \cdots + \frac{n-1}{2}\right) \\
&= 2\left(\frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2} + 1\right)\right) \\
&= \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \\
&= \frac{n^2-1}{4}
\end{aligned}$$

When n is even,

$$\begin{aligned}
s(n) &= 0 + 1 + 1 + 2 + 2 + \cdots + \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 1\right) + \frac{n}{2} \\
&= \left(1 + 2 + \cdots + \left(\frac{n}{2} - 1\right)\right) + \left(1 + 2 + \cdots + \left(\frac{n}{2} - 1\right) + \frac{n}{2}\right) \\
&= \left(\frac{1}{2}\left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right)\right) + \left(\frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right)\right) \\
&= \left(\frac{1}{2}\left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right)\right) + \left(\frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n+2}{2}\right)\right) \\
&= \left(\frac{n^2-2n}{8}\right) + \left(\frac{n^2+2n}{8}\right) \\
&= \frac{n^2}{4}
\end{aligned}$$

Thus,

$$s(n) = \begin{cases} \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \\ \frac{n^2}{4} & \text{if } n \text{ is even.} \end{cases}$$

(e) Suppose that m and n are any two positive integers with $m > n$. Prove that $s(m+n) - s(m-n) = mn$.

Solution:
Note that $m+n$ and $m-n$ differ by $2n$, which is even. Thus, $m+n$ and $m-n$ are either both odd or both even. In the case when they are both odd,

$$\begin{aligned} s(m+n) - s(m-n) &= \frac{(m+n)^2 - 1}{4} - \frac{(m-n)^2 - 1}{4} \\ &= \frac{(m^2 + 2mn + n^2 - 1) - (m^2 - 2mn + n^2 - 1)}{4} \\ &= \frac{4mn}{4} \\ &= mn \end{aligned}$$

Meanwhile, in the case when they are both even,

$$\begin{aligned} s(m+n) - s(m-n) &= \frac{(m+n)^2}{4} - \frac{(m-n)^2}{4} \\ &= \frac{(m^2 + 2mn + n^2) - (m^2 - 2mn + n^2)}{4} \\ &= \frac{4mn}{4} \\ &= mn \end{aligned}$$

Suppose that A is an $n \times n$ matrix such that every entry of A is ± 1. Show that the determinant of A is divisible by 2^{n-1}.

Solution:
We will give a proof by induction. For the base case, we will consider the case when $n = 1$. Then $A = (1)$ or $A = (-1)$, so $\det(A)$ equals 1 or -1, both of which are divisible by $2^{n-1} = 2^0 = 1$.

For the inductive hypothesis, assume that for some $k \geq 1$, the determinant of any $k \times k$ matrix whose entries are all ± 1 is divisible by 2^{k-1}.

For the inductive step, suppose that A is any $(k+1) \times (k+1)$ matrix such that every entry of A is ± 1. Let B be the $(k+1) \times (k+1)$ matrix obtained from A by replacing row 1 of A by the sum of row 1 with row 2. Since each entry of A is 1 or -1, the entries in the first row of B must all be 2, 0, or -2, while the entries in the remaining rows of B must all be 1 or -1. Moreover, $\det(A) = \det(B)$ since the type of elementary row operation performed does not change the determinant. Therefore, it suffices to show that $\det(B)$ is divisible by $2^{(k+1)-1} = 2^k$.

By using cofactor expansion along the first row of B, we see that

$$\det(B) = \sum_{j=1}^{k+1} (-1)^{1+j} b_{1j} \det(M_{1j}),$$

where M_{1j} is the $k \times k$ matrix obtained from B by removing row 1 and column j. For each j, M_{1j} is a $k \times k$ matrix whose entries are all 1 or -1, so by the inductive hypothesis, $\det(M_{1j})$ is divisible by 2^{k-1} for all j. Moreover, for each j, b_{1j} is divisible by 2 since b_{1j} equals 2, 0, or -2. Thus, for each j, $b_{1j} \det(M_{1j})$ is divisible by $2 \cdot 2^{k-1} = 2^k$, so

$$\det(B) = \sum_{j=1}^{k+1} (-1)^{1+j} b_{1j} \det(M_{1j})$$

is also divisible by 2^k. Since $\det(A) = \det(B)$, $\det(A)$ is divisible by 2^k, completing our induction.

A robot is programmed to shuffle cards in such a way so that it always rearranges cards in the same way relative to the order in which the cards are given to it. The thirteen hearts arranged in the order

$$A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$$

are given to the robot, shuffled, and then the shuffled cards are given back to the robot and shuffled again. This process is repeated until the cards have been shuffled a total of 7 times. If at this point the order of the cards is

$$2, 4, 6, 8, 10, Q, A, K, J, 9, 7, 5, 3,$$

what was the order of the cards after the first shuffle?

Solution:
Let σ denote the permutation implemented by the robot. Then in cycle notation,

$$\sigma^7 = (A\ 7\ J\ 9\ 10\ 5\ Q\ 6\ 3\ K\ 8\ 4\ 2).$$

Thus, $|\sigma^7| = 13$. We claim that this implies that $|\sigma| = 13$ as well. To see this, note that $|\sigma^7| = 13$ implies that $(\sigma^7)^{13} = e$, where e denotes the identity permutation. Then $\sigma^{91} = e$. Thus implies that $|\sigma|$ divides 91, meaning that $|\sigma|$ is 1, 7, 13, or 91. But $|\sigma|$ cannot equal 1 or 7 because then σ^7 would have to equal e, which it doesn't. Moreover, if $|\sigma|$ was 91, then the least common multiple of the lengths of the cycles when σ is written in disjoint cycle notation would have to be 91, which would mean that σ would have to have disjoint cycles of length 7 and 13 or a cycle of length 91, both of which are impossible with only a total of 13 cards to permute. Therefore, σ must have order 13. Since $\sigma^{13} = e$, $(\sigma^7)^2 = \sigma^{14} = \sigma\sigma^{13} = \sigma e = \sigma$. Hence, in cycle notation,

$$\sigma = (\sigma^7)^2 = (A\ J\ 10\ Q\ 3\ 8\ 2\ 7\ 9\ 5\ 6\ K\ 4),$$

so the order of the cards after the first shuffle is

$$4, 8, Q, K, 9, 5, 2, 3, 7, J, A, 10, 6.$$

Suppose A is a non-empty, closed[1] subset of $\mathbb{R}$ such that for each $a \in A$, every open interval that contains a also contains another element of A. Show that A must be uncountable.

Solution:
First note that A cannot be finite because if it was, then around each element a of A we would be able to find an open interval small enough so that it contains a but no other element of A, contradicting the given condition on A. Thus, A must be infinite.

Next, we will show that A cannot be countably infinite via contradiction. Suppose that A is countably infinite. Then the elements of A can be listed out as $a_1, a_2, a_3, \dots$. Let U_1 be the open interval $(a_1 - 1, a_1 + 1)$ around a_1. By the condition on A, this open interval must contain another element in A; without loss of generality, assume that a_2 is in U_1 (if not, relabel the elements of A). Next, let U_2 be a small enough open interval around a_2 so that the closure of U_2 (i.e. the closed interval consisting of U_2 and the corresponding endpoints) is contained inside U_1 and so that a_1 is not in the closure of U_2. Again, by the condition on A, U_2 must contain another element of A; without loss of generality, assume that a_3 is in U_2. Now choose an open interval U_3 around a_3 such that the closure of U_3 is contained in U_2 and so that neither a_1 nor a_2 is contained in the closure of U_3. Without loss of generality, assume that a_4 is in U_3. In general, for each $n \geq 2$, find a small enough open interval U_n around a_n such that

- the closure of U_n is contained in U_{n-1}
- $a_1, a_2, \dots, a_{n-1}$ are not contained in the closure of U_n
- a_{n+1} is contained in U_n (after a possible relabeling of the points in $\{a_{n+1}, a_{n+2}, \dots\}$)

For each $n \geq 1$, let $V_n = \text{closure}(U_n) \cap A$. Note that each V_n is closed and bounded, hence compact. Moreover, $V_1 \supset V_2 \supset V_3 \supset \cdots$. Let $V = \cap_{i=1}^{\infty} V_n$. Since V is the intersection of a decreasing nested sequence of non-empty compact subsets of A, V is a non-empty subset of A. However, none of the a_n's can be in V since for $i \geq n$, a_n is not in V_i. Since we've arrived at a contradiction, A cannot be countably infinite. Thus, it must be the case that A is uncountably infinite.

[1] A is closed in $\mathbb{R}$ if and only if $\mathbb{R} - A$ consists of the union of any number of open intervals.