Spring 2016 Indiana Collegiate Mathematics Competition (ICMC) Exam

COMPLETE QUESTIONS AND SOLUTIONS

Mathematical Association of America – Indiana Section

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(1) Assume we are looking at numbers in normal decimal representations, and consider the set $A = \{1, 11, 111, 1111, \ldots\} = \{x \in \mathbb{Z} \mid x \text{ consists entirely of 1s}\}$. For all $a \in A$, define $n(a)$ to be the number of 1s in a's representation. (For example, $n(1) = 1$ and $n(111) = 3$.) Prove or disprove the following statement: The set $\{a \in A \mid n(a) \text{ divides } a\}$ is infinite.

Solution: To prove that the set is infinite, we will use induction to show that for $a \in A$, $n(a)|a$ whenever a consists of 3^m ones (i.e. $n(a) = 3^m$) for any integer $m \geq 0$.

For the base case when $m = 0$, note that if $n(a) = 3^0 = 1$, $a = 1$, so $n(a)|a$. Now assume that for some integer $k \geq 0$, $n(b)$ divides b when b is the element of A consisting of 3^k ones, and consider the element c of A consisting of 3^{k+1} ones. Note that $b \cdot 10^{3^k}$ consists of 3^k ones followed by 3^k zeros, while $b \cdot 10^{2 \cdot 3^k}$ consists of 3^k ones followed by $2 \cdot 3^k$ zeros. Therefore, c can be built from 3 copies of b via the formula

$$\begin{aligned} c &= b + b \cdot 10^{3^k} + b \cdot 10^{2 \cdot 3^k} \\ &= b(1 + 10^{3^k} + 10^{2 \cdot 3^k}). \end{aligned}$$

Note that $3^k = n(b)$ divides b by the inductive hypothesis, and 3 divides $d = (1 + 10^{3^k} + 10^{2 \cdot 3^k})$ since the sum of the digits of d is 3. Therefore, $3^k \cdot 3 = 3^{k+1} = n(c)$ divides c, which completes the proof.

(2) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function with the following properties:
(i) $f(2) = 2$
(ii) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$
(iii) $f(m) > f(n)$ whenever $m > n$
Conjecture what all the possible candidates for f are, and then prove your conjecture.

Solution: f must be the identity function. In other words, $f(n) = n$ for all $n \in \mathbb{Z}$.

Proof. First, we will use strong induction to show that $f(n) = n$ for all integers $n \geq 2$. For the base case, note that $f(2) = 2$ is property (i) of f. Now assume that for some integer $k \geq 2$, $f(n) = n$ for all integers $2 \leq n \leq k$, and consider $f(k + 1)$.

- Case 1: If $k + 1$ is composite, then $k + 1$ can be written as the product of two integers m and n where $2 \leq m \leq k$ and $2 \leq n \leq k$. Then

$$\begin{aligned} f(k + 1) &= f(mn) \\ &= f(m)f(n) \text{ by property (ii) of f} \\ &= mn \text{ by the inductive hypothesis} \\ &= k + 1 \end{aligned}$$

- Case 2: If $k+1$ is prime, then $k+2$ must be composite since $k \geq 2$, so $k+2$ can be written as the product of two integers m and n, where $2 \leq m \leq k+1$ and $2 \leq n \leq k+1$. We can actually lower the upper bound on both m and n from $k+1$ to k since $\frac{k+2}{k+1} < 2$ when $k \geq 2$. Then similarly to above, $f(k+2) = f(mn) = f(m)f(n) = mn = k+2$. Since $f(k) = k$ (by the inductive hypothesis) and $f(k+2) = k+2$, we then know that $f(k+1) = k+1$ by property (iii) of f.

So far we have shown that $f(n) = n$ for all integers $n \geq 2$. Next, note that

$$\begin{aligned} f(0) &= f(2 \cdot 0) \\ &= f(2)f(0) \\ &= 2f(0), \end{aligned}$$

so it must be the case that $f(0) = 0$. Since $f(0) = 0$ and $f(2) = 2$, we know that $f(1) = 1$ by property (iii) of f. Also, $f(-1) = -1$ because $f(-1) < f(0) = 0$ and

$$\begin{aligned} (f(-1))^2 &= f(-1)f(-1) \\ &= f(-1 \cdot -1) \\ &= f(1) \\ &= 1. \end{aligned}$$

To complete the proof, we need to show that $f(n) = n$ for negative integers $n \leq -2$. Note that if n is a negative integer, then $-n$ is a positive integer, so we know that $f(-n) = -n$ by our previous work. Then

$$\begin{aligned} f(n) &= f(-1 \cdot -n) \\ &= f(-1) \cdot f(-n) \\ &= -1 \cdot -n \\ &= n. \end{aligned}$$

Thus, for all integers n, $f(n) = n$, so f must be the identity function. $\square$

(3) Arrange 8 points in 3-space so that each of the 56 possible triplets of points determined forms an isosceles triangle. Prove your arrangement works. (Note: Some triangles may be degenerate!)

Solution: One such solution would be the points

$$\begin{array}{ll} (0,0,0) & (1,0,0) \\ (\cos(2\pi/5), \sin(2\pi/5), 0) & (\cos(4\pi/5), \sin(4\pi/5), 0) \\ (\cos(6\pi/5), \sin(6\pi/5), 0) & (\cos(8\pi/5), \sin(8\pi/5), 0) \\ (0,0,1) & (0,0,-1) \end{array}$$

Geometrically, these points represent the origin, the five vertices of a regular pentagon in the xy-plane with each point distance 1 away from the origin, and the two points on the line perpendicular to the xy-plane through the origin distance 1 away from the origin. To see that any choice of three points gives you an isosceles triangle, consider the following cases:

Case 1: One of the three points is the origin. Then by construction the other two points are each distance one away from the origin, and so two sides of the triangle are congruent. Note that when the other two points are the points off of the xy-plane, the triangle is degenerate.

Case 2: All three points are on the pentagon. Then either one point is a vertex adjacent to the other two vertices on the pentagon, or one point is a vertex opposite the other two vertices on the pentagon. In either case, the distance between that vertex and the other vertices is the same, giving us again an isosceles triangle.

Case 3: One of the vertices is outside the xy-plane, the two other vertices are on the pentagon. Then by the Pythagorean theorem, both points on the pentagon are distance $\sqrt{2}$ away from the third vertex, which gives us an isosceles triangle.

Case 4: One vertex is on the pentagon, the other two are off the xy-plane. Then the point on the plane is $\sqrt{2}$ away from the points off the plane, which again gives us an isosceles triangle.

(4) Let A be a subset of $\mathbb{R}$, and let $f, g : A \to A$ be two continuous functions. Then f is said to be homotopic to g if there is a continuous function $h : A \times [0,1] \to A$ such that $h(a, 0) = f(a)$ and $h(a, 1) = g(a)$ for all a in A. (This function is sometimes called a deformation function. It may be helpful to think of $[0, 1]$ as being a "slider" that continuously morphs the function f into the function g.)

(a) Let $A = [0, 1] \subset \mathbb{R}$, and define $f, g : A \to A$ by $f(a) = a^2$ and $g(a) = a^3$. Find a function h demonstrating that f and g are homotopic.

(b) Recall that an equivalence relation $\sim$ on a set X requires three properties:
- *Reflexivity*: $\forall\, x \in X,\ x \sim x$
- *Symmetry*: $\forall\, x, y \in X$, if $x \sim y$, then $y \sim x$
- *Transitivity*: $\forall\, x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$

Prove that for any subset A of $\mathbb{R}$, being homotopic forms an equivalence relation on the set $\mathcal{C}(A)$ of all continuous functions $f : A \to A$.

Solution:

(a) Define $h : A \times [0, 1] \to A$ by $h(a, t) = (1-t)f(a) + tg(a) = (1-t)a^2 + ta^3$. Note that $h(a, t) \in A = [0, 1]$ for all $a \in A = [0, 1]$ and all $t \in [0, 1]$. Moreover, h is a continuous function such that $h(a, 0) = (1 - 0)f(a) + 0g(a) = f(a)$ and $h(a, 1) = (1 - 1)f(a) + 1g(a) = g(a)$, so f and g are homotopic.

(b)
- *Reflexivity*: Suppose $f : A \to A$ is any continuous function on A. Define $h : A \times [0, 1] \to A$ by $h(a, t) = f(a)$. Then h is a continuous function such that $h(a, 0) = f(a)$ and $h(a, 1) = f(a)$ for all $a \in A$, so f is homotopic to f.
- *Symmetry*: Suppose $f, g : A \to A$ are any two continuous functions on A such that f is homotopic to g. This implies that there exists

a continuous function $h : A \times [0,1] \to A$ such that $h(a,0) = f(a)$ and $h(a,1) = g(a)$ for all $a \in A$. Define $H : A \times [0,1] \to A$ by $H(a,t) = h(a, 1-t)$. Then H is a continuous function such that $H(a,0) = h(a,1-0) = h(a,1) = g(a)$ and $H(a,1) = h(a,1-1) = h(a,0) = f(a)$ for all $a \in A$. Therefore, g is homotopic to f.

- *Transitivity*: Suppose that $f, g, i : A \to A$ are any three continuous functions on A such that f is homotopic to g and g is homotopic to i. This implies that there exist continuous functions $h : A \times [0,1] \to A$ and $h' : A \times [0,1] \to A$ such that $h(a,0) = f(a)$, $h(a,1) = g(a)$, $h'(a,0) = g(a)$, and $h'(a,1) = i(a)$ for all $a \in A$. Since $h(a,1) = g(a) = h'(a,0)$ for all $a \in A$, the function $H : A \times [0,1] \to A$ defined by

$$H(a,t) = \begin{cases} h(a,2t) & \text{if } 0 \le t \le \frac{1}{2} \\ h'(a,2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

is continuous, and $H(a,0) = h(a, 2\cdot 0) = h(a,0) = f(a)$ and $H(a,1) = h'(a, 2\cdot 1 - 1) = h'(a,1) = i(a)$ for all $a \in A$. Therefore, f is homotopic to i.

(5) The Fibonacci numbers f_n $(n = 0,1,2,\dots)$ are defined recursively by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-2} + f_{n-1}$ for $n \ge 2$.
(a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(f_n, f_{n+1}) = (f_{n+1}, f_{n+2})$ for all $n \ge 0$, and then state and prove a conjecture for what $T^n(0,1)$ equals for all $n \ge 1$.
(b) Find the matrix A of T with respect to the standard basis for $\mathbb{R}^2$, and then find the eigenvalues of A.
(c) Find a non-recursive expression for f_n for all $n \ge 1$.

Solution:
(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (y, x+y)$ is a linear transformation such that $T(f_n, f_{n+1}) = (f_{n+1}, f_n + f_{n+1}) = (f_{n+1}, f_{n+2})$ for all integers $n \ge 0$. We will prove using induction that $T^n(0,1) = (f_n, f_{n+1})$ for all integers $n \ge 1$. For the base case, note that $T^1(0,1) = T(0,1) = (1, 0+1) = (1,1) = (f_1, f_2)$. Now assume that $T^k(0,1) = (f_k, f_{k+1})$ for some integer $k \ge 1$, and consider $T^{k+1}(0,1)$. Note that

$$\begin{aligned} T^{k+1}(0,1) &= T(T^k(0,1)) \\ &= T(f_k, f_{k+1}) \text{ by the inductive hypothesis} \\ &= (f_{k+1}, f_k + f_{k+1}) \\ &= (f_{k+1}, f_{k+2}). \end{aligned}$$

Thus, we can conclude that $T^n(0,1) = (f_n, f_{n+1})$ for all integers $n \ge 1$.
(b) Since $T(1,0) = (0, 1+0) = (0,1)$ and since $T(0,1) = (1, 0+1) = (1,1)$, the standard matrix A of T is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

.

The characteristic polynomial of A is $(0-\lambda)(1-\lambda)-1 = \lambda^2-\lambda-1$, so the eigenvalues of A are $\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$. Note that $\lambda_1 = \varphi$ is the golden ratio, while $\lambda_2 = 1-\varphi$.

(c) The vector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \varphi \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = \varphi = \frac{1+\sqrt{5}}{2}$ since

$$\begin{aligned} A\mathbf{v}_1 &= \begin{bmatrix} \varphi \\ 1+\varphi \end{bmatrix} \\ &= \varphi \begin{bmatrix} 1 \\ 1+\frac{1}{\varphi} \end{bmatrix} \\ &= \varphi \begin{bmatrix} 1 \\ \varphi \end{bmatrix} \\ &= \varphi \mathbf{v}_1, \end{aligned}$$

and similarly, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1-\varphi \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 1-\varphi$. Note that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} (\mathbf{v}_1 - \mathbf{v}_2).$$

Therefore,

$$\begin{aligned} A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{\sqrt{5}} (\varphi^n \mathbf{v}_1 - (1-\varphi)^n \mathbf{v}_2) \\ &= \frac{1}{\sqrt{5}} \left(\varphi^n \begin{bmatrix} 1 \\ \varphi \end{bmatrix} - (1-\varphi)^n \begin{bmatrix} 1 \\ 1-\varphi \end{bmatrix} \right) \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^n - (1-\varphi)^n \\ \varphi^{n+1} - (1-\varphi)^{n+1}. \end{bmatrix} \end{aligned}$$

Since A is the standard matrix of T, this implies that

$$T^n(0,1) = \frac{1}{\sqrt{5}} \left(\varphi^n - (1-\varphi)^n, \varphi^{n+1} - (1-\varphi)^{n+1} \right).$$

By part (a), the first coordinate of $T^n(0,1)$, namely $\frac{1}{\sqrt{5}} (\varphi^n - (1-\varphi)^n)$, equals f_n. This provides us our desired non-recursive expression for f_n:

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - (1-\varphi)^n).$$

(6) Let A be an open subset of the real numbers and $f : A \to \mathbb{R}$ be a differentiable function such that $f'(a) = 0$ for all $a \in A$.

(a) Prove that if $A = \mathbb{R}$, f must be a constant function. (Note: Do not use antiderivatives in your proof, as the proof that antiderivatives are unique up to a constant depends on this statement.)

(b) Prove or disprove: If $A \neq \mathbb{R}$, f must be a constant function.

Solution:

(a) Let a, b be any two real numbers with $a < b$. By the Mean Value Theorem, there exists a real number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f'(c) = 0$, this implies that $0 = \frac{f(b)-f(a)}{b-a}$, so that $f(a)$ must equal $f(b)$. Since a and b are arbitrary real numbers, f must be a constant function.

(b) To see that this is false, suppose that $A = (0, 1) \cup (2, 3)$, and define $f : A \to \mathbb{R}$ by

$$f(a) = \begin{cases} 1 & \text{if } 0 < a < 1 \\ 2 & \text{if } 2 < a < 3. \end{cases}$$

Then $f'(a) = 0$ for all a in A, but f is not a constant function.

(7) Let G be a set and $* : G \times G \to G$ be a binary operation. If $*$ satisfies the following conditions:

- *Associativity*: $\forall\, a, b, c \in G$, $(a * b) * c = a * (b * c)$
- *Identity*: $\exists e \in G$ such that $\forall\, g \in G$, $e * g = g * e = g$
- *Inverse*: $\forall\, a \in G$, $\exists\, a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

we say G is a group. For purposes of shorthand, the symbol $*$ is often omitted (i.e., we write $a * b$ as simply ab and $a * a * a$ as simply a^3).

Assume that G is a group which has the following additional properties:

(i) For all $g, h \in G, (gh)^3 = g^3h^3$.

(ii) There are no elements in G of order 3 (i.e., $\nexists\, g \in G$ such that $g^3 = e$).

For any group G with properties (i) and (ii) listed above:

(a) Prove that for all $a, b \in G$, if $a \neq b$, then $a^3 \neq b^3$.

(b) Prove that the map $\phi : G \to G$ defined by $\phi(g) = g^3$ is a bijection if G is finite.

Solution:

(a) Assume to the contrary that a and b are two elements of such a group G such that $a \neq b$ but $a^3 = b^3$. Note that $(b^3)^{-1} = (b^{-1})^3$ because

$$\begin{aligned} b^3(b^{-1})^3 &= (bb^{-1})^3 \\ &= e^3 \\ &= e \end{aligned}$$

Therefore, $a^3 = b^3$ implies that

$$\begin{aligned} a^3(b^3)^{-1} &= b^3(b^3)^{-1} \\ a^3(b^{-1})^3 &= e \\ (ab^{-1})^3 &= e \text{ by property (i) of G} \end{aligned}$$

Since there are no elements of G of order 3 (by property (ii) of G) and $(ab^{-1})^3 = e$, it must be the case that $ab^{-1} = e$, which implies that $a = b$, which contradicts our assumption that $a \neq b$. Therefore, it must be the case that $a^3 \neq b^3$ whenever $a \neq b$.

(b) Assume that G contains n elements and that $\phi : G \to G$ is defined by $\phi(g) = g^3$. Note that the previous part implies that ϕ is one-to-one since $\phi(a) = a^3$ equals $\phi(b) = b^3$ if and only if $a = b$. We will show that ϕ must be onto via contradiction. Assume that ϕ is not onto. Then there exists $g \in G$ such that g is not in the image of ϕ. Then the image of the n elements of G must lie inside the set $G - \{g\}$ that only contains $(n-1)$ elements. Then the pigeonhole principle implies that there must exist some $g' \in G - \{g\}$ and two distinct elements a, b in G such that $\phi(a) = g' = \phi(b)$. However, this is impossible since ϕ is one-to-one. Therefore, it must be the case that ϕ is onto. Since ϕ is one-to-one and onto, ϕ is a bijection.

(8) (**Note: This is a puzzle problem. The length is for clarity, not necessarily its difficulty.**) Assume elections are conducted by voters who place all the candidates in rank order. Under these conditions there are several possible voting methods available:

- Plurality – the candidate with the most first-place votes wins.
- Plurality with Elimination – the following process is repeated until there's a winner:
  - If there's a candidate that has more than half the first-place votes, he/she wins.
  - Otherwise, eliminate the candidate(s) with the fewest first-place votes. Reorder the voters' preferences by moving other candidates up in the ranks to fill the vacancies.
- Borda count – candidates receive points based upon each voter's ranking. A last-place vote earns a candidate one point; a second-to-last-place vote earns a candidate two points; a third-to-last-place vote earns a candidate three points; and so forth, with a first-place vote earning a candidate n points, where n is the total number of candidates. The candidate with the most total points wins.
- Pairwise Comparison – candidates receive points based upon their performance in a round-robin style analysis. Each candidate is paired with each other candidate. (For example, if there were four candidates, there would be six possible matchups.) For each matchup, count the number of voters who prefer one candidate over the other. The winner of the matchup (more voter preferences) receives a point; if there's a tie, both candidates receive half a point. The candidate with the most total points wins.
- Survivor – the following process is repeated until there's a winner:
  - If there's only one candidate remaining, he/she wins.
  - Otherwise, eliminate the candidate(s) with the most last-place votes. Reorder the voters' preferences by moving other candidates up in the ranks to fill the vacancies.

Example: Consider the following sample preference table for 9 voters in an election with 3 candidates: X, Y, and Z:

Rank/Number of Votes	4	3	2
First	X	Y	Z
Second	Y	Z	Y
Third	Z	X	X

That is, 4 voters rank X as their first choice, Y as their second choice, and Z as their third choice; 3 voters would pick Y first, Z second and X third; and 2 voters would choose Z first, Y second, and X third. Using this table of preferences the winners for the various voting methods would be:

- Plurality – X wins with the most first-place votes (4 of them)
- Plurality with Elimination – no candidate has more than half the first place votes, so we eliminate the candidate(s) with the fewest. Candidate Z is eliminated. The new preference table is

Rank/Number of Votes	4	3	2
First	X	Y	Y
Second	Y	X	X

  Now Y wins with more than half the first-place votes.
- Borda count – the point totals are as follows:
  - X: 4(3)+3(1)+2(1)=17
  - Y: 4(2)+3(3)+2(2)=21
  - Z: 4(1)+3(2)+2(3)=16

  Therefore, Y wins.
- Pairwise Comparison – the matchups are as follows:
  - X vs. Y: Y wins 5 to 4 because 5 voters like Y better than X; Y gets a point
  - X vs. Z: Z wins 5 to 4 because 5 voters like Z better than X; Z gets a point
  - Y vs. Z: Y wins 7 to 2 because 7 voters like Y better than Z; Y gets a point

  Therefore, Y wins.
- Survivor – we eliminate the candidate(s) with the most last-place votes. Candidate X is eliminated. The new preference table is

Rank/Number of Votes	4	3	2
First	Y	Z	Y
Second	Z	Y	Z

  Now Z is eliminated, leaving only Y remaining; therefore, Y wins.

Challenge: Construct a preference table for an election with five candidates – A, B, C, D, and E – such that A wins via plurality, B wins via plurality with elimination, C wins via Borda count, D wins via pairwise comparison, and E wins via survivor. You may use any number of voters you would like.

Solution: Consider the following sample preference table for 27 voters in an election with 5 candidates: A, B, C, D, and E:

Rank/Number of Votes	2	6	7	8	4
First	C	D	B	A	C
Second	E	C	E	D	B
Third	D	E	C	E	E
Fourth	A	B	D	C	D
Fifth	B	A	A	B	A

- Plurality – A wins with the most first-place votes (8 of them)
- Plurality with Elimination – no candidate has more than half the first place votes, so we eliminate the candidate(s) with the fewest first place votes. Candidate E is eliminated. The new preference table is

Rank/Number of Votes	2	6	7	8	4
First	C	D	B	A	C
Second	D	C	C	D	B
Third	A	B	D	C	D
Fourth	B	A	A	B	A

Still, no candidate has more than half the first place votes, so we eliminate the candidate(s) with the fewest. Candidates C and D are eliminated. The new preference table is

Rank/Number of Votes	2	6	7	8	4
First	A	B	B	A	B
Second	B	A	A	B	A

B now has more than half of the first place votes, so B is the winner.

- Borda count – the point totals are as follows:
  - A: 2(2)+6(1)+7(1)+8(5)+4(1)=61
  - B: 2(1)+6(2)+7(5)+8(1)+4(4)=73
  - C: 2(5)+6(4)+7(3)+8(2)+4(5)=91
  - D: 2(3)+6(5)+7(2)+8(4)+4(2)=90
  - E: 2(4)+6(3)+7(4)+8(3)+4(3)=90

  Therefore, C wins.
- Pairwise Comparison – the relevant matchups are as follows:
  - D vs. A: D wins 19 to 7 because 19 voters like D better than A;
  - D vs. B: D wins 16 to 11 because 16 voters like D better than B;
  - D vs. C: D wins 14 to 13 because 14 voters like D better than C;
  - D vs. E: D wins 14 to 13 because 14 voters like D better than E;

  Therefore, since D went undefeated and untied, and since all other candidates lost at least once (to D), D wins.
- Survivor – we eliminate the candidate(s) with the most last-place votes. Candidate A is eliminated. The new preference table is

Rank/Number of Votes	2	6	7	8	4
First	C	D	B	D	C
Second	E	C	E	E	B
Third	D	E	C	C	E
Fourth	B	B	D	B	D

Now B is eliminated, so the new preference table is

Rank/Number of Votes	2	6	7	8	4
First	C	D	E	D	C
Second	E	C	C	E	E
Third	D	E	D	C	D

Now D is eliminated, so the new preference table is

Rank/Number of Votes	2	6	7	8	4
First	C	C	E	E	C
Second	E	E	C	C	E

Now C is eliminated, so the E wins.