Spring 2015 Indiana Collegiate Mathematics Competition (ICMC) Exam

COMPLETE QUESTIONS AND SOLUTIONS

Mathematical Association of America – Indiana Section

Written by: Brian Rice (Huntington University)
Edited by: Justin Gash (Franklin College) and Feng Tian (Trine University)

(1) Say that an integer n has a super-3 representation if there is a positive integer m, a sequence of distinct nonnegative integers $p_1, \dots, p_m$, and a sequence $a_1, \dots, a_m$ where each a_k is ± 1, so that

$$n = \sum_{k=1}^{m} a_k \cdot 3^{p_k} = a_1 \cdot 3^{p_1} + \cdots + a_m \cdot 3^{p_m}.$$

For instance, the integer 8 has the super-3 representation $8 = 3^2 - 3^0$ and the integer -11 has the super-3 representation $-11 = -3^2 - 3^1 + 3^0$. The number 0 has the *empty* super-3 representation; i.e., where $m = 0$ and the sum has no terms.

(a) Give a super-3 representation of 2015.
(b) Prove that every integer n has a super-3 representation.

Solution:

(a) The number 2015 has the super-3 representation $2015 = 3^7 - 3^5 + 3^4 - 3^2 - 3^0$.

(b) Without loss of generality we may let n be a nonnegative integer, since given a super-3 representation of n, we can find a super-3 representation of $-n$ by changing the signs of all the a_k. Now, we prove this by strong induction on n. As given in the problem statement, 0 has a super-3 representation; this is our base case. Now, let $n > 0$ and suppose that every $0 \le x < n$ has a super-3 representation. Then n is either a multiple of 3, one more than a multiple of 3, or one less than a multiple of 3. Let q be the nearest multiple of 3 to n, and let $r = q/3$. Certainly $0 \le r < n$, so by the inductive step it has a super-3 representation; say $r = \sum_{k=1}^{m} a_k \cdot 3^{p_k}$. Then $q = \sum_{k=1}^{m} a_k \cdot 3^{p_k+1}$; then either $n = q$, and that sum is a representation of n, or $n = q \pm 1$. In the latter case,

- if $n = q + 1$, then $n = \sum_{k=1}^{m} a_k \cdot 3^{p_k+1} + 3^0$ (so $a_{m+1} = 1$ and $p_{m+1} = 0$)
- if $n = q - 1$, then $n = \sum_{k=1}^{m} a_k \cdot 3^{p_k+1} - 3^0$ (so $a_{m+1} = -1$ and $p_{m+1} = 0$)

In any case this gives a super-3 representation of n since the exponents of 3 are all distinct. Each positive exponent is one more than its corresponding exponent in a known super-3 representation (thus all positive exponents are distinct), and if one of the exponents is 0, it is the only such exponent. This proves that n has a super-3 representation, completing the inductive step and the proof.

Comment: Several other solutions are possible, including different induction arguments and a construction based on base-3 representations.

(2) Prove that, for every positive integer n,

$$\sum_{k=0}^{n} \sum_{i=0}^{n-k} \binom{n}{k} \binom{n-k}{i} 2^{k+i} = 5^n.$$

First Solution: Rewrite the left hand side as

$$\sum_{k=0}^{n}\binom{n}{k}2^k\sum_{i=0}^{n-k}\binom{n-k}{i}2^i.$$

Applying the binomial theorem to the interior sum we obtain $(2+1)^{n-k} = 3^{n-k}$, so we find that the left hand side is equal to

$$\sum_{k=0}^{n}\binom{n}{k}2^k3^{n-k}.$$

Applying the binomial theorem again, we find that the left hand side is equal to $(2+3)^n = 5^n$ as required.

Second Solution: The term inside the double sum counts the number of ways to choose, from a group of n people, two disjoint committees A and B of sizes k and i respectively, and then to choose a (possibly empty) third committee C from among the members of both committees A and B. The double sum then adds all these up over all possible values of k and i. Hence the sum as a whole counts the number of ways to choose from n people two disjoint committees A and B, and then a third committee C from among the members of A and B. We can count this number in a different way: Each person can be either (1) on no committees at all, (2) on committee A only, (3) on committee B only, (4) on committees A and C, or (5) on committees B and C. Since each person independently has 5 possible assignments, the total number of ways to do this is 5^n, as required.

(3) Three pairwise perpendicular line segments $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ have endpoints all on a sphere of unknown radius, and intersect inside the sphere at a point X. Given the lengths $AX = 1$, $CX = 2$, $EX = 3$, and $BX = 4$, determine, with proof, the volume of the octahedron with vertices A, B, C, D, E, and F.

Solution: Since $\overline{AB}$ and $\overline{CD}$ intersect, the quadrilateral $ACBD$ is planar and inscribed in the circle that its plane cuts from the sphere. By the power of a point theorem, $AX \cdot BX = CX \cdot DX$, whence $1 \cdot 4 = 2 \cdot DX$ and $DX = 2$. Since $\overline{AB}$ and $\overline{CD}$ are perpendicular, the area N of $ACBD$ is $N = \frac{1}{2}AB \cdot CD = \frac{1}{2}(5 \cdot 4) = 10$.

The same argument shows that $1 \cdot 4 = AX \cdot BX = EX \cdot FX = 3 \cdot FX$, and thus $FX = \frac{4}{3}$. Since $\overline{EF}$ is perpendicular to the plane containing $ACBD$, the octahedron is divided by that plane into two right pyramids with base area $N = 10$ and heights $EX = 3$ and $FX = \frac{4}{3}$ respectively. These two pyramids have volumes $\frac{1}{3}EX \cdot N = \frac{1}{3}(3 \cdot 10) = 10$ and $\frac{1}{3}FX \cdot N = \frac{1}{3}(\frac{4}{3} \cdot 10) = \frac{40}{9}$. Hence the volume of the octahedron is $10 + \dfrac{40}{9} = \dfrac{130}{9}$.

Comment: The argument above can be used to prove the little-known fact that the power of a point theorem is true in three dimensions as well as in two.

(4) You are standing in a room, which we will call Σ, which contains eight light switches numbered 1 through 8, all in the off position. On the other side of the door is another room, Λ, which contains eight lights, labeled A through H. Each switch controls exactly one light. Your goal is to determine which switches control which lights. You do this by making a number of trials, which consist of putting some set of switches in room Σ in the on position, then entering room Λ to discover which lights are on. For instance, one trial might be to turn switches 1 and 3 to the on position (and all others off) and observe which lights in room Λ are on (perhaps lights D and H, though of course you can't know this ahead of time).

(a) Give a strategy for finding out which switches control which lights using the smallest possible number of trials. (For this part, you do not need to prove that the number of trials you use is minimal.)

(b) Prove that your strategy uses the minimal number of trials; that is, prove that there is no strategy that determines which switches control which lights using fewer trials.

Solution:

(a) The minimum number of trials is 3. In the first trial, turn on switches 1, 2, 3 and 4; in the second, switches 1, 2, 5, and 6, and in the third, 1, 3, 5, and 7. Then the light that was on in all three trials is that controlled by switch 1, that which is on in trials 1 and 2, but not trial 3, is controlled by switch 2, and so on. It is easy to verify that each light is on in a difference subset of the trials, so this determines which switches control which lights.

(b) Observe that if two switches x and y are always on or off together in our trials, then our trials cannot distinguish which of them control which light. For instance, if in every trial, switches 1 and 2 are either both on or both off, and these switches control lights A and B, we will not be able to determine whether switch 1 controls light A or light B. Therefore, if we can use a set of trials to determine which switches control which lights, it must be the case that the function from switches to subsets of tests given by $f(x) = \{\alpha \mid \text{Light } x \text{ is on in Test } \alpha\}$ is one-to-one. Since the number of subsets of all trials is 2 to the power of the number of trials, and the number of switches is 8, it follows that there must be at least 3 trials.

Comment: The same argument can be used to show that in the general case of n switches, we need to use at least $\lceil \log_2(n) \rceil$ trials to determine which switches control which lights.

(5) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called C^∞ if all of its derivatives exist everywhere.

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ function with infinitely many zeros in the interval $[0, 1]$. Show that there is some $x \in [0, 1]$ such that $f^{(n)}(x) = 0$ for every integer $n \geq 0$ (that is, such that f and all its derivatives vanish at x).

(b) Give an example of such a function f which is nonconstant on every interval. (You need not prove that your function works.)

Solution:

(a) Let $a_1, a_2, \ldots$ be an infinite sequence of zeros of f in the interval $[0,1]$. Since every bounded infinite sequence has a convergent subsequence, we may find such a convergent subsequence $b_1, b_2, \ldots$. Let $x = \lim_{n\to\infty} b_n$. Certainly, $x \in [0,1]$. Now, there are either infinitely many b_n less than x or infinitely many b_n greater than x; without loss of generality, suppose the former. It then follows that we may choose an infinite increasing sequence $c_1, c_2, \ldots$ from among the b_n; simply begin with $c_1 < x$, and continue choosing c_{n+1} from among the b_i, such that $c_n < c_{n+1} < x$ (We can always do this because otherwise we would have infinitely many b_i which are less than c_n, which is impossible since $\lim_{n\to\infty} b_n = x > c_n$.) Since this increasing sequence is a subsequence of the b_n, it necessarily converges to x.

Since f is differentiable, it is continuous, and it follows that $f(x) = f(\lim_{n\to\infty} c_n) = \lim_{n\to\infty} f(c_n) = 0$. Now by Rolle's theorem, there is a zero of f' between c_n and c_{n+1} for every $n \geq 1$; let $c_n^{(1)}$ be a zero of f' between c_n and c_{n+1}. Then the sequence $c_1^{(1)}, c_2^{(1)}, \ldots$ is an increasing sequence of zeros of f', and by the squeeze theorem it has limit x. As f' is itself differentiable and thus continuous, it follows as above that $f'(x) = 0$. Continuing by induction, we find that for every n, there is an increasing sequence $c_1^{(n)}, c_2^{(n)}, \ldots$ of zeros of $f^{(n)}$ with limit x, and hence that $f^{(n)}(x) = 0$.

This shows that our chosen x satisfies the required condition.

(b) One such function is given by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Comment: The function given above for the second part is a modification of the classic example of a nonzero function with identically zero Taylor series at 0.

(6) Recall that a graph is called *planar* if it can be drawn on the plane in such a way that no two of its edges cross. Further recall that a graph is c-colorable (for a positive integer c) if its vertices can be colored with c colors in such a way that no two adjacent vertices have the same color. The famous *Four Color Theorem* says that every planar graph can be 4-colored.

Call a graph k-color-planar if it can be drawn on the plane, and its *edges* colored with k colors, in such a way that no two edges with the same color cross. Thus, a 1-color-planar graph is just a planar graph.

Prove that, for every positive integer k, there is a positive integer c_k such that every k-color-planar graph is c_k-colorable. (You may use the Four Color Theorem in your proof if you wish.)

Solution: Let G be a k-color-planar graph. Each of the k sets of edges with the same color forms a planar graph on the vertices of G. By the Four Color

Theorem, each of these graphs can be 4-colored. Doing so, each vertex of G gets a k-tuple of colors corresponding to its color in each of the k planar graphs. If we assign a different color to each k-tuple, this gives us a 4^k-coloring of G (since adjacent vertices must differ in at least one coordinate as they are adjacent in some one of the k planar graphs). Thus, every k-color-planar graph can be 4^k-colored (so $c_k = 4^k$ works).

Comment: Is this value $c_k = 4^k$ best possible? We don't know and would be interested in a proof either way.

(7) Let S be a finite set and $* : S \to S$ be a binary operation on S. Suppose that $*$ satisfies the following two conditions:
- $*$ is associative; that is, for any $a, b, c \in S$, $(a * b) * c = a * (b * c)$.
- For any $a, b \in S$, $a * (a * (b * a)) = b$.

(a) Prove that $*$ is commutative; that is, that for any $a, b \in S$, $a * b = b * a$.
(b) Prove that $|S|$, the size of S, is a power of 3.

Solution:

(a) Because $*$ is associative we may omit parentheses without any ambiguity. Let $a, b \in S$ be arbitrary. We have

$$\begin{aligned} a * b &= a * (a * a * b * a) \\ &= a * a * a * (a * a * b * a) * a \\ &= a * (a * a * a * a) * b * a * a \\ &= a * a * b * a * a \\ &= b * a \end{aligned}$$

where each line uses the fact that $a * a * b * a = b$ (and in the case $a = b$, $a * a * a * a = a$). This shows that $*$ is commutative.

(b) It follows that, for any $a, b \in S$, $b * (a * a * a) = (a * a * a) * b = b$, so $(a*a*a)$ acts as an identity for $*$. The identity is unique, since if e and f are both identities, then $e = e * f = f$. Call the identity e. Furthermore, since $a * a * a = e$ for every $a \in S$, it follows that $a * a$ is an inverse for a. Thus, $*$ is an associative binary operation with an identity and inverses, and therefore $(S, *)$ is a group.
Moreover, the fact that $a*a*a = e$ for every $a \in S$ implies that the order of every element of S divides 3. It follows by Cauchy's theorem that $|S|$ cannot be divisible by any prime other than 3, so $|S|$ is a power of 3.

Comment: In the last step, the fundamental theorem of finite abelian groups can be used in place of Cauchy's theorem.

(8) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $[0, 1]$, and suppose that $f(0) = f'(0) = 0$ and $f(1) = 1$. Prove that there is some $a \in (0, 1)$ such that $f'(a)f''(a) = \dfrac{9}{8}$.

Solution: Define $g(x) = f(x) - x^{\frac{3}{2}}$. Then $g(0) = 0$ and $g(1) = 0$, so by the mean value theorem, there is some $c \in (0,1)$ such that $g'(c) = 0$ and hence $f'(c) = \dfrac{3}{2}\sqrt{c}$. Thus $(f'(c))^2 = \dfrac{9}{4}c$. Applying the mean value theorem to $(f')^2$ on the interval $[0,c]$, we obtain that there is some $a \in (0,c)$ (and hence in $(0,1)$) such that $2f'(a)f''(a) = ((f')^2)'(a) = \dfrac{9}{4}$. Thus $f'(a)f''(a) = \dfrac{9}{8}$ as required.