Spring 2014 Indiana Collegiate Mathematics Competition (ICMC) Exam

COMPLETE QUESTIONS AND SOLUTIONS

Mathematical Association of America – Indiana Section

Written by: The Mathematics Faculty of Indiana University - Purdue University Fort Wayne

Edited by: Justin Gash, Franklin College

(1) Let $a > 0$, and define the following function:

$$f(x) = \frac{\sqrt{a^3x} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}.$$

- Calculate these limits:

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \\ \lim_{x\to a} f(x) &= \\ \lim_{x\to +\infty} f(x) &= \end{aligned}$$

- Find the maximum value of $f(x)$ on its domain.

Solution: The function

$$f(x) = \frac{a^{3/2}x^{1/3}(x^{1/6} - a^{1/6})}{a^{1/4}(a^{3/4} - x^{3/4})}$$

is continuous on $[0, a)$, with the $x \to 0^+$ limit equal to 0.

For the $x \to a$ limit, the $\frac{0}{0}$ form of L'Hôpital's Rule applies.

$$\begin{aligned}\lim_{x\to a} \frac{\sqrt{a^3x} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} &= \lim_{x\to a} \frac{a^{3/2}x^{1/2} - a^{5/3}x^{1/3}}{a - a^{1/4}x^{3/4}} \\ (LHR) &= \lim_{x\to a} \frac{a^{3/2}\frac{1}{2}x^{-1/2} - a^{5/3}\frac{1}{3}x^{-2/3}}{0 - a^{1/4}\frac{3}{4}x^{-1/4}} \\ &= \frac{\frac{1}{2}a - \frac{1}{3}a}{-\frac{3}{4}} \\ &= -\frac{2a}{9}\end{aligned}$$

For the $x \to +\infty$ limit, LHR could be used again, but it is easier to notice f satisfies $|f(x)| < Cx^{-1/4}$ for large x, so there is a horizontal asymptote $f(x) \to 0$ as $x \to \infty$.

From the above expression, $f(x) < 0$ for all $x \in (0, a) \cup (a, \infty)$, so the maximum value is $f(0) = 0$.

Comment: You may try to find critical points for the second part, but this is a difficult calculation and a waste of time.

(2) Let f be a function with domain $(0, \infty)$ satisfying:

- $f(x) = f(x^2)$ for all $x > 0$
- $\lim_{x\to 0^+} f(x) = \lim_{x\to +\infty} f(x) = f(1)$

Show that $f(x)$ is a constant function on $(0, \infty)$.

Solution: For integer $k \geq 1$, $f(x^{(2^k)}) = f((x^{(2^{k-1})})^2) = f(x^{(2^{k-1})})$, so by induction, $f(x^{(2^k)}) = f(x)$ for all integer $k \geq 0$. Given $\varepsilon > 0$, there is some $\delta \in (0, 1)$ and some $N \in (1, \infty)$ so that if $0 < t < \delta$ or $t > N$, then $|f(t) - f(1)| < \varepsilon$. If $0 < x < 1$, then there is some integer k such that

$k > \log_2(\ln(\delta)/\ln(x))$, which is equivalent to $0 < x^{(2^k)} < \delta$, and if $x > 1$, then there is some integer k such that $k > \log_2(\ln(N)/\ln(x))$, which is equivalent to $N < x^{(2^k)}$, so in either case, $|f(x) - f(1)| = |f(x^{(2^k)}) - f(1)| < \varepsilon$. Since ε was arbitrary, $f(x) = f(1)$.

Comment: You may try using more informal limit arguments, but at the risk of taking some unjustified steps.

(3) Let V be a corner of a right-angled box and let x, y, z be the angles formed by the long diagonal and the face diagonals starting at V. For

$$A = \begin{bmatrix} \sin x & \sin y & \sin z \\ \sin z & \sin x & \sin y \\ \sin y & \sin z & \sin x \end{bmatrix}$$

show that $|\det(A)| \leq 1$.

Solution: From $\sin = \frac{opp}{hyp}$, $\sin x = a/d$, $\sin y = b/d$, and $\sin z = c/d$, where a, b, c are the side lengths of the box and d is the long diagonal length.

$$\det(A) = \frac{1}{d^3} \det \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

The absolute value of

$$\det \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

is the volume of a parallelepiped with side lengths all equal to d. By the scalar triple product formula, such a volume is maximized when the parallelepiped has all right angles, so it is a cube with volume d^3. The claimed inequality follows.

Comment: Is there some less geometric approach, maybe an obscure inequality comparing $\det = a^3 + b^3 + c^3 - 3abc$ to $d^3 = (a^2 + b^2 + c^2)^{3/2}$? The authors would be interested to know.

(4) Let $f(t)$ be a real valued integrable function on $[0, 1]$, so that both sides of the following equation are continuous functions of x:

$$2x - 1 = \int_0^x f(t)dt.$$

Prove that if $f(t) \leq 1$ for $0 \leq t \leq 1$, then there exists a unique solution $x \in [0, 1]$ of the equation.

Solution: Let $F(x)$ be the function $2x-1-\int_0^x f(t)dt$, which is continuous on $[0,1]$ and satisfies $F(0)=-1$ and $F(1)=1-\int_0^1 f(t)dt \geq 1-\int_0^1 1dt = 0$. By the Intermediate Value Theorem, $F(x)=0$ has at least one solution $x\in[0,1]$. This solution is unique because F is increasing on $[0,1]$: for $0\leq a<b\leq 1$,

$$F(b)-F(a) = 2(b-a)-\int_a^b f(t)dt \geq 2(b-a)-1(b-a) = b-a > 0$$

Comment: If $f(t)$ were continuous, then F could be proved increasing using the Fundamental Theorem of Calculus: $F'(x)=2-f(x)\geq 1$. However, the problem specifically omits this hypothesis.

(5) Let $ABCD$ be a rectangle. The bisector of the angle ACB intersects AB at point M and divides the rectangle $ABCD$ into two regions: the triangle MBC with area s and the convex quadrilateral $MADC$ with area t.

- Determine the dimensions of the rectangle $ABCD$ in terms of s and t.
- If $t=4s$, what is the ratio AB/BC?

Solution: Let $AB=b$, $BC=h$, $AM=y$, $MB=x$, and let θ be half the angle ABC, and let α be the angle BMC. By the Law of Sines,

$$\frac{\sin\theta}{y} = \frac{\sin(\pi-\alpha)}{AC},\quad \frac{\sin\alpha}{h} = \frac{\sin\theta}{x} \implies \frac{\sin\alpha}{\sin\theta} = \frac{h}{x} = \frac{AC}{y} = \frac{\sqrt{b^2+h^2}}{y}$$

We have the following system of polynomial equations.

$$\begin{aligned} x+y &= b \\ \frac{1}{2}xh &= s \\ bh &= s+t \\ x^2(b^2+h^2) &= h^2y^2 \end{aligned}$$

Eliminating y first gives:

$$x^2(b^2+h^2) = h^2(b-x)^2 \implies x^2b = h^2b - 2h^2x$$

Multiplying both sides by h^3 gives:

$$\begin{aligned} x^2bh^3 &= h^4(hb-2hx) \\ (2s)^2(s+t) &= h^4(s+t-2(2s)) \\ h &= \left(\frac{4s^2(s+t)}{t-3s}\right)^{1/4} \\ b &= \frac{s+t}{h} = \frac{(s+t)^{3/4}(t-3s)^{1/4}}{\sqrt{2s}} \end{aligned}$$

The b/h ratio can be computed directly for $t=4s$, or as:

$$\frac{b}{h} = \frac{bh}{h^2} = \frac{s+4s}{\left(\frac{4s^2(s+4s)}{4s-3s}\right)^{1/2}} = \frac{5s}{\sqrt{20s^2}} = \frac{\sqrt{5}}{2}$$

Comment: The equality of ratios $\frac{h}{x} = \frac{AC}{y}$ from the first step is also known as the "bisector theorem" for triangles.

(6) In a badly overcrowded pre-school, every child is either left-handed or right-handed, either blue-eyed or brown-eyed, and either a boy or a girl. Exactly half of the children are girls, exactly half of the children are left-handed and exactly one fourth of the children are both. There are twenty-six children who are brown-eyed. Nine of those twenty-six are right-handed boys. Two children are right-handed boys with blue eyes. Thirteen children are both left-handed and brown-eyed. Five of these thirteen are girls.

- How many students does the pre-school have?
- How many girls are right-handed and blue-eyed?

Solution: There are 8 types of students with the following populations:

$$\begin{aligned}
\#\text{ RH BL boy} &= 2 \\
\#\text{ RH BR boy} &= 9 \\
\#\text{ LH BR girl} &= 5 \\
\#\text{ LH BR boy} &= 13-5=8 \\
\#\text{ RH BR girl} &= 26-9-13=4 \\
\#\text{ LH BL girl} &= x \\
\#\text{ RH BL girl} &= y \\
\#\text{ LH BL boy} &= z
\end{aligned}$$

From equal numbers of boys and girls, $x+y+9=19+z$. From equal numbers of LH and RH, $x+z+13=y+15$. From one fourth LH girls, $4(x+5)=x+y+z+28$. This is a system of three linear equations in three unknowns. Standard solution methods give the unique answer $x=6$, $y=7$, and $z=3$, so the total population is $x+y+z+28=44$, with 7 RH BL girls.

Comment: Drawing a Venn diagram may be helpful.

(7) Let $n>1$ be an integer. Let $(G,\cdot)$ be a group, with an identity element e and an element $a\in G$ with $a\neq e$ and $a^n=e$. Let $(H,*)$ be a group, let $f:G\to H$ be an arbitrary function, and then define $F:G\to H$ by:

$$F(x)=f(x)*f(a\cdot x)*f(a^2\cdot x)*\ldots*f(a^{n-1}\cdot x)$$

- Show that if $f(G)$ is a subset of some Abelian subgroup of H, then F is not a one-to-one function.
- Let $(H,*)$ be the symmetric group $(S_3,\circ)$ (the six-element group of permutations of three objects). Give an example of $(G,\cdot)$, n, and a as above, and a function $f:G\to H$, so that the expression F is a one-to-one function.

Solution: For the first part,

$$\begin{aligned}
F(e) &= f(e)*f(a\cdot e)*f(a^2\cdot e)*\ldots*f(a^{n-1}\cdot e) \\
&= f(e)*f(a)*f(a^2)*\ldots*f(a^{n-1})
\end{aligned}$$

$$\begin{aligned} F(a) &= f(a) * f(a^2) * f(a^2 \cdot a) * \ldots * f(a^{n-2} \cdot a) * f(a^{n-1} \cdot a) \\ &= f(a) * f(a^2) * f(a^3) * \ldots * f(a^{n-1}) * f(e) = F(e) \end{aligned}$$

using the property that $f(e)$ commutes with other $f(g)$ at the last step. By the assumption that $a \neq e$, F is not one-to-one.

For the second part, there are lots of examples. (A correct answer must have explicit examples of G, n, a, and f.) A simple one is to let G be a two element group $\{e, a\}$, so $n = 2$, and to define $f : G \to S_3$ by $f(e) = (12)$ and $f(a) = (23)$, or any other pair of non-commuting elements in S_3. Then $F(e) = f(e) * f(a) = (12) \circ (23) = (123)$ and $F(a) = f(a) * f(e) = (23) \circ (12) = (132)$, so F is one-to-one.

(8) Determine whether the following sum of real cube roots is rational or irrational:

$$\sqrt[3]{6 + \sqrt{\frac{847}{27}}} + \sqrt[3]{6 - \sqrt{\frac{847}{27}}}$$

Solution: Let x be the number; then a short calculation with convenient cancellations shows x satisfies $x^3 = 12 + 5x$. The only real root of $x^3 - 5x - 12 = (x - 3)(x^2 + 3x + 4)$ is $x = 3$.

Comment: This is similar to problem #3 from the 1969 ICMC.