Spring 2013 Indiana Collegiate Mathematics Competition (ICMC) Exam

COMPLETE QUESTIONS AND SOLUTIONS

Mathematical Association of America – Indiana Section

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(1) Define a sequence (s_n) recursively as follows: Let $s_1 = 1$ and for $n \geq 1$, let $s_{n+1} = \sqrt{1 + s_n}$. Prove that (s_n) converges, and then find the limit.

Solution: We show that the sequence is monotone increasing and bounded above. It then follows from the Monotone Convergence Theorem that the sequence converges. We show that for every $n \in \mathbb{N}$, we have $s_n < 2$ and $s_{n+1} \geq s_n$. This is done by induction.

For $n = 1$, we have $s_1 = 1 < 2$ and $s_2 = \sqrt{2} \geq 1 = s_1$.

Assume that for some $n \in \mathbb{N}$, we have $s_n < 2$ and $s_{n+1} \geq s_n$. Then $s_{n+1} = \sqrt{1 + s_n} < \sqrt{1 + 2} < 2$. Also, $s_{n+2} = \sqrt{1 + s_{n+1}} \geq \sqrt{1 + s_n} = s_{n+1}$.

By the principle of Mathematical Induction, we obtain that every $n \in \mathbb{N}$, we have $s_n < 2$ and $s_{n+1} \geq s_n$. Because the sequence is bounded above and monotone increasing, it converges.

Let $s = \lim_{n\to\infty} s_n$. We observe that $s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \sqrt{1 + s_n} = \sqrt{1 + s}$. Solving the equation $s = \sqrt{1 + s}$ for s yields $s = \dfrac{1 + \sqrt{5}}{2}$. □

(2) Let $\mathcal{C}$ be a non-empty collection (possibly infinite) of compact subsets of $\mathbb{R}$.
  (a) Prove that $K = \bigcap_{C \in \mathcal{C}} C$ is a compact set.
  (b) Give an example that illustrates that the union of a family of compact sets need not be compact.

Solution:
  (a) We let $C_0 \in \mathcal{C}$ be an arbitrary element of the collection. Because C_0 is compact, it is closed and bounded, by the Heine-Borel Theorem. Also, $K = \bigcap_{C \in \mathcal{C}} C \subseteq C_0$, so K is bounded. Each member C of the family $\mathcal{C}$ is compact and therefore closed. Therefore, the intersection $K = \bigcap_{C \in \mathcal{C}}$ is closed.
  Because K is closed and bounded, it is compact by the Heine-Borel Theorem. □
  (b) Let $C_n = [-n, n]$ for all $n \in \mathbb{N}$. Each set C_n is closed and bounded, and is therefore compact. Let $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$. Then $\bigcup_{C \in \mathcal{C}} C = \mathbb{R}$, which is not compact (it is not bounded). □

(3) Assume A and B are two sets with m and n elements, respectively.
  (a) How many one-to-one functions are there from A and B?
  (b) How many one-to-one and onto functions are there from A to B?

Solution:
  (a) We distinguish two cases.

(i) If $m > n$, there are no one-to-one functions from A to B by the pigeonhole property.

(ii) If $m \leq n$, then the principle of multiplication shows there are nPm or $P(n,m) = \frac{n!}{(n-m)!} = n \cdot (n-1) \cdots (n-m+1)$ one-to-one functions from A to B. Equivalently, if one enumerates the elements in A as $(a_i)_{i=1}^{m}$ and the elements of B as $(b_j)_{j=1}^{n}$, there are clearly n available images for domain element a_1; without loss of generality, say this element is b_1. If one is to maintain the one-to-one property, there are now $n-1$ available images for a_2; without loss of generality, say the image is b_2. This pattern continues, yielding $n-2$ possible images for a_3, $n-3$ images for a_4, and so on. This also yields the answer $n \cdot (n-1) \cdots (n-m+1)$.

(b) If m is not equal to n there is no one-to-one and onto function from A to B. If m is equal to n then by the principle of multiplication there are $n!$ one-to-one and onto functions from A to B. □

(4) Let p and q be distinct prime numbers. Find the number of generators of the group $\mathbb{Z}_{pq}$.

Solution: An element $a \in \mathbb{Z}_{pq}$ is a generator of $\mathbb{Z}_{pq}$ if and only if a and pq are relatively prime. Because p and q are primes, the elements of $\mathbb{Z}_{pq}$ that are not relatively prime to pq are the multiples of p and the multiples of q. The multiples of p are $p, 2p, \ldots, (q-1)p$ (i.e., $q-1$ multiples). Using a similar argument, we see that there are $p-1$ multiples of q. Also, 0 is not a generator. Therefore, there are $(q-1)+(p-1)+1 = p+q-1$ elements of $\mathbb{Z}_{pq}$ that are not generators, leaving $pq-p-q+1$ elements that are generators. □

(5) Let G be a group and H a subgroup of G with index $(G : H) = 2$. Prove that H is a normal subgroup of G.

Solution:
Because the index of H in G is 2, there are exactly two left cosets of H in G, and there are exactly two right cosets. The left cosets are H itself, and a coset of the form aH, for some $a \in G$. Likewise, the right cosets are H and a set of the form Hb for some $b \in G$. Observe that $H \cap aH = \emptyset$ and $H \cup aH = G$; likewise $H \cap Hb = \emptyset$ and $H \cup Hb = G$. It follows that $aH = Hb$, which means that the left and the right cosets of H coincide, making H a normal subgroup of G. □

(6) The Fibonacci numbers are defied as

$$f_1 = f_2 = 1$$

and

$$f_{n+1} = f_n + f_{n-1}$$

for $n \geq 3$.

(a) List $f_1, f_2, \ldots, f_7$.

(b) Illustrate, using the list from (a), that $f_{2n+1} = f_{n+1}^2 + f_n^2$ for $n = 1, 2, 3$.

(c) Prove that $f_{2n+1} = f_{n+1}^2 + f_n^2$ for all $n \in \mathbb{N}$.

Solution:

(a)

n	1	2	3	4	5	6	7
f_n	1	1	2	3	5	8	13

(b) For $n = 1$, we observe $f_{2(1)+1} = f_3 = 2$ and $f_{1+1}^2 + f_1^2 = (1)^2 + (1)^2 = 2$.
For $n = 2$, we observe $f_{2(2)+1} = f_5 = 5$ and $f_{2+1}^2 + f_2^2 = (2)^2 + (1)^2 = 5$.
For $n = 3$, we observe $f_{2(3)+1} = f_7 = 13$ and $f_{3+1}^2 + f_3^2 = (3)^2 + (2)^2 = 13$.

(c) For $n = 1$ and $n = 2$, the formula has been verified in part (b). Therefore, the basis steps hold for mathematical induction. Now assume, for the strong form of mathematical induction, the identity holds for all values of n up to $n = k - 1$. Then

$$f_{2k-3} = f_{k-1}^2 + f_{k-2}^2$$

and

$$f_{2k-1} = f_k^2 + f_{k-1}^2$$

Now we need to verify that the identity holds for $n = k$. In order to do this, we calculate f_{2k+1}.

$$\begin{aligned} f_{2k+1} &= f_{2k} + f_{2k-1} \\ &= f_{2k-1} + f_{2k-2} + f_{2k-1} \\ &= 2f_{2k-1} + (f_{2k-1} - f_{2k-3}) \\ &= 3f_{2k-1} - f_{2k-3} \end{aligned}$$

Substituting the induction hypothesis, we can write the last expression as

$$\begin{aligned} f_{2k+1} &= 3\left(f_k^2 + f_{k-1}^2\right) - f_{k-1}^2 - f_{k-2}^2 \\ &= 3f_k^2 + 2f_{k-1}^2 - (f_k - f_{k-1})^2 \\ &= 2f_k^2 + f_{k-1}^2 + 2f_k f_{k-1} \\ &= 2f_k^2 + (f_{k+1} - f_k)^2 + 2f_k(f_{k+1} - f_k) \\ &= 2f_k^2 + (f_{k+1} - f_k)(f_{k+1} - f_k + 2f_k) \\ &= 2f_k^2 + (f_{k+1} - f_k)(f_{k+1} + f_k) \\ &= f_{k+1}^2 + f_k^2 \end{aligned}$$

This completes the induction step. □

(7) Let a, b, m, M be real numbers with $0 < m \leq a \leq b \leq M$, prove that

$$\frac{2\sqrt{mM}}{m + M} \leq \frac{2\sqrt{ab}}{a + b}$$

Solution: Since all numbers are positive, it is sufficient to prove

$$\frac{4mM}{(m + M)^2} \leq \frac{4ab}{(a + b)^2}$$

This is equivalent to proving

$$\frac{4\frac{m}{M}}{\left(1+\frac{m}{M}\right)^2} \le \frac{4\frac{a}{b}}{\left(1+\frac{a}{b}\right)^2}$$

Consider the function $f(x) = \dfrac{4x}{(1+x)^2}$. This function is increasing on $[0,1]$. To see this we note that $\dfrac{d}{dx}\left(\dfrac{4x}{(1+x)^2}\right) = 4\dfrac{1-x}{(x+1)^3} \ge 0$ on $[0,1]$.

From $0 < m \le a$, and $M > 0$ we obtain $0 < \dfrac{m}{M} \le \dfrac{a}{M}$. Because $b \le M$, we also have $\dfrac{a}{M} \le \dfrac{a}{b}$. Finally, $a \le b$, so $\dfrac{a}{b} \le 1$. Therefore, $0 < \dfrac{m}{M} \le \dfrac{a}{b} \le 1$.

Hence we have

$$f\left(\frac{m}{M}\right) = \frac{4\frac{m}{M}}{\left(1+\frac{m}{M}\right)^2} \le f\left(\frac{a}{b}\right) = \frac{4\frac{a}{b}}{\left(1+\frac{a}{b}\right)^2}$$

□

(8) A soccer ball is stitched together using white hexagons and black pentagons. Each pentagon borders five hexagons. Each hexagon borders three other hexagons and three pentagons. Each vertex is of valence 3 (meaning that at each corner of a hexagon or pentagon, exactly three hexagons or pentagons meet). How many hexagons and how many pentagons are needed to make a soccer ball? **Hint:** Euler's Polyhedron Formula states that $V - E + F = 2$, where V is the number of vertices, E is the number of edges (i.e., the line adjoining two vertices) and F is the number of faces (hexagons or pentagons).

Solution: We let F_5 denote the number of pentagons, F_6 denote the number of hexagons. We consider the soccer ball to be a polyhedron, with $F = F_5 + F_6$ faces, E edges and V vertices. By Euler's Formula $V - E + F = 2$.

Each vertex is of valence 3. We may think of placing an observer on each vertex, and let the observers report the number of faces they see. Each observer reports seeing 3 faces. Each face is observed by as many observers as there are corners on the face, so we obtain $3V = 5F_5 + 6F_6$.

Now place an observer in every hexagon and let them report the number of pentagons that border their hexagon. There are F_6 observers, each reporting 3 pentagons, for a total of $3F_6$ reports. Each pentagon is bordered by 5 hexagons, so each pentagon be reported by 5 different observers, so $3F_6 = 5F_5$.

Now place an observer into each of the faces and let them report the number of edges they see. Each edge will be observed by two observers, so $2E = 5F_5 + 6F_6$.

Beginning with the equation $V - E + F = 2$ (and multiplying by 6) we obtain $6V - 6E + 6F = 12$. Substituting $3V = 5F_5 + 6F_6$ and $2E = 5F_5 + 6F_6$ and $F = F_5 + F_6$ we obtain $2(5F_5 + 6F_6) - 3(5F_5 + 6F_6) + 6(F_5 + F_6) = 12$, or $F_5 = 12$.

From the equation $3F_6 = 5F_5$ we get $F_6 = 20$. Therefore, the soccer ball has 12 black pentagons and 20 white hexagons. □