Spring 2012 Indiana Collegiate Mathematics Competition (ICMC) Exam

COMPLETE QUESTIONS AND SOLUTIONS

Mathematical Association of America – Indiana Section

Written by: Ahmed Mohammed, Ball State University
Edited by: Justin Gash, Franklin College

(1) Show that n^2 divides $(n+1)^n - 1$ for any positive integer n.

Solution:

Obviously, the statement is true for $n = 1$. So we assume that $n \geq 2$. By the Binomial Theorem we note that

$$
\begin{aligned}
(n+1)^n - 1 &= \sum_{j=0}^{n} \binom{n}{j} n^j - 1 \\
&= \sum_{j=1}^{n} \binom{n}{j} n^j \\
&= \binom{n}{1} n + \sum_{j=2}^{n} \binom{n}{j} n^j = n^2 + \sum_{j=2}^{n} \binom{n}{j} n^j
\end{aligned}
$$

On noting that

$$\binom{n}{j}$$

is a positive integer, we see that

$$\sum_{j=2}^{n} \binom{n}{j} n^j$$

is divisible by n^2. Therefore, indeed $(n+1)^n - 1$ is divisible by n^2. $\square$

(2) How many zeros are at the end of $213\,!$?

Solution:

First we observe that the number of zeros at the end of $213\,!$ is same as the number of times the number 10 occurs as a factor of $213\,!$. Since $10 = 2 \times 5$, and since there are more factors of 2 than 5 in the digit $213\,!$ we observe that there are as many zeros in the product $213\,!$ as there are 5's in the product $213\,!$. In other words there are as many zeros at the end of $213\,!$ as there are 5's in $213\,!$. So we proceed to count the number of times 5 occurs as a factor in the product $213\,!$. The number of positive multiples of 5 less or equal to 213 is

$$\left[\frac{213}{5} \right] = 42 \tag{*}$$

where $[x]$ denotes the greatest integer less than or equal to x. Among these multiples we list those that contain two or more factors of 5 as follows.

$$25 = 5^2, \quad 50 = 2 \cdot 5^2, \quad 75 = 3 \cdot 5^2, \quad 100 = 4 \cdot 5^2,$$

$$125 = 5^3, \quad 150 = 6 \cdot 5^2, \quad 175 = 7 \cdot 5^2, \quad 200 = 8 \cdot 5^2$$

Therefore there are 9 additional occurrences of the digit 5 that have not been counted in (*). Therefore there are a total of $42+9 = 51$ occurrences of the digit in the product $213\,!$. Hence there are 51 zeros at the end of $213\,!$. $\square$

(3) Let $p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial of degree $n \geq 2$, with integer coefficients, such that a_0, a_1, a_n and $a_2 + \cdots + a_n$ are odd integers. Show that $p(x)$ has no rational root. Give example to show that the conclusion may not be true if any of a_0, a_1, a_n or $a_2 + \cdots + a_n$ is even.

Solution:

Suppose r/s is a rational root so that r and s are integers with $s \neq 0$. Clearly $r \neq 0$. We suppose that $\gcd(r, s) = 1$. We immediately see that both r and s can't be even. We proceed to show that all other parities lead to a contradiction. By assumption we have

$$0 = p\left(\frac{r}{s}\right) = a_n \left(\frac{r}{s}\right)^n + \cdots + a_2 \left(\frac{r}{s}\right)^2 + a_1 \left(\frac{r}{s}\right) + a_0$$

Clearing fractions we see that

$$a_n r^n + a_{n-1} s r^{n-1} + \cdots + a_2 s^{n-2} r^2 + a_1 s^{n-1} r + a_0 s^n = 0 \tag{0.1}$$

Suppose now r is even and s is odd. This would imply that

$$a_n r^n + a_{n-1} s r^{n-1} + \cdots + a_2 s^{n-2} r^2 + a_1 s^{n-1} r$$

is even. Therefore we conclude from (0.1) that $a_0 s^n$ is even as well. But this is not possible, since $a_0 s^n$ is odd as a_0, and s (hence s^n) are both odd. Recalling that a_n is odd, a similar argument shows that r odd and s even is not possible either.

We now show that s and r can't be both odd. To this end, we add the odd integer $a_n + \cdots + a_2$ to both sides of (0.1) and get

$$a_n(1+r^n) + a_{n-1}(1+sr^{n-1}) + \cdots + a_2(1+s^{n-2}r^2) + a_1 s^{n-1} r + a_0 s^n = a_n + \cdots + a_2 \tag{0.2}$$

Suppose now both s and r are odd. Then $1 + s^i r^j$ is an even integer for any non-negative integers i, j. Therefore $a_n(1+r^n) + a_{n-1}(1+sr^{n-1}) + \cdots + a_2(1+s^{n-2}r^2)$ is even. Since a_0 and a_1 are both odd, we note that each of $a_1 s^{n-1} r$ and $a_0 s^n$ is odd, and therefore their sum is even. Therefore we see that the

left hand side in (0.2) is even. But we recall that the right hand side is odd. Thus, once again, we conclude that r and s can't be both odd.
Let $p(x) = a_2x^2 + a_1x + a_0$. We consider three cases: $a_2 = 2,\ a_1 = a_0 = -1$ or $a_2 = a_0 = -1,\ a_1 = 2$ or $a_2 = a_1 = -1,\ a_0 = 2$. Note that in each case $p(1) = 0$. This example shows that none of the conditions in the problem may be omitted. □

(4) Let A be an $n \times n$ matrix whose diagonal entries are all equal to the same real number $\alpha \in \mathbb{R}$ and all other entries are equal to $\beta \in \mathbb{R}$. Show that A is diagonalizable, and compute the determinant of A.

Solution:

First let us dispose of the trivial case when $\beta = 0$. In this case $A = \alpha I_n$ which is obviously diagonalizable, and $\det A = \alpha^n$. So, henceforth we suppose that $\beta \neq 0$.
Note that $A = \vec{w}\vec{v}^T - \gamma I_n$, where

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} \beta \\ \beta \\ \vdots \\ \beta \end{bmatrix} \qquad \text{and} \qquad \gamma = \beta - \alpha$$

Now, a non-zero vector $\vec{x}$ in $\mathbb{R}^n$ is an eigenvector of A if and only if $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$. That is

$$\vec{w}\vec{v}^T\vec{x} - \gamma\vec{x} = \lambda\vec{x}$$

We rewrite this as

$$(\vec{v}^T\vec{x})\,\vec{w} = (\lambda + \gamma)\,\vec{x}$$

If $c = \vec{v}^T\vec{x} \neq 0$, then note that $\lambda + \gamma \neq 0$, and therefore

$$\vec{x} = c(\lambda + \gamma)^{-1}\,\vec{w},$$

showing that $\vec{x}$ is a multiple of $\vec{w}$. If $\vec{v}^T\vec{x} = 0$, then $\vec{x}$ is orthogonal to $\vec{v}$ and hence to $\vec{w}$ (recall that $\beta \neq 0$). Thus, any eigenvector of A is either a multiple of $\vec{w}$ or orthogonal to $\vec{w}$. Therefore we see that $\vec{w}$ is an eigenvector of A with

$$A\vec{w} = \vec{w}\vec{v}^T\vec{w} - \gamma\vec{w} = (n\beta - \gamma)\,\vec{w} = ((n-1)\beta + \alpha)\,\vec{w}$$

That is $\lambda = (n-1)\beta + \alpha$ is an eigenvalue of A with corresponding eigenspace of dimension 1. On the other hand, any other eigenvector of A must be orthogonal to $\vec{w}$ with corresponding eigenvalue $\lambda = -\gamma = \alpha - \beta$. The eigenspace of A corresponding to $\lambda := \alpha - \beta$ is the orthogonal complement of the eigenspace of A corresponding to the eigenvalue $\lambda = (n-1)\beta + \alpha$, namely the orthogonal complement of the line parallel to $\vec{w}$. Thus the eigenspace of A corresponding to the eigenvalue $\lambda = (n-1)\beta + \alpha$ has dimension $n-1$. This shows that A is diagonalizable. The determinant of A is the product of its eiegnevalues

$$\det A = (\alpha - \beta)^{n-1}[(n-1)\beta + \alpha)]$$

□

(5) Let G be a group of order 26. If G has a normal subgroup of order 2, show that G is a cyclic group.

Solution:

Let $N = \{e, a\}$ be a normal subgroup of G of order 2, where e is the identity of the group. It follows that a has order 2. Since N is ia a normal subgroup of G, by definition, we see that $g^{-1}ag \in G$ for any $g \in G$. Since $a \neq e$ we must have $g^{-1}ag = a$. That is $ag = ga$. Hence we have shown that $ag = ga$ for all $g \in G$. Since $ag = ga$, one can easily show by the principle of mathematical induction that $(ag)^n = a^n g^n$ for any $g \in G$, and any non-negative integer n. The quotient group G/N is of order 13, and since any group of prime order is cyclic, this quotient group is cyclic. Let $b \in G \setminus N$. Then the coset bN has order 13. In particular the order of b can't be 2. Since its order has to divide 26, the order of b must be either 13 or 26. If the order is 26, then G is cyclic with generator b. If the order is 13, then, since $ab = ba$ we must have $(ab)^{13} = a^{13}b^{13} = a$, and therefore $(ab)^{26} = a^2 = e$, and hence ab has order 26. Therefore ab generates G, and therefore G is cyclic. □

(6) Show that for any positive integer k, the following is an irrational number.

$$\sum_{n=0}^{\infty} \frac{1}{(n\,!)^k}$$

Solution:

Suppose contrary to what is asserted, the indicated sum is a rational number. Then there are positive integers a and b such that

$$\sum_{n=0}^{\infty} \frac{1}{(n\,!)^k} = \frac{a}{b}$$

We multiply both sides by $((b+1)\,!)^k$ and rewrite the sum as follows.

$$\begin{aligned}((b+1)\,!)^k\frac{a}{b} &= \sum_{n=0}^{\infty}\frac{((b+1)\,!)^k}{(n\,!)^k}\\ &= \sum_{n=0}^{b+1}\frac{((b+1)\,!)^k}{(n\,!)^k} + \sum_{n=b+2}^{\infty}\frac{((b+1)\,!)^k}{(n\,!)^k}\end{aligned}$$

Thus

$$\sum_{n=b+2}^{\infty}\frac{((b+1)\,!)^k}{(n\,!)^k} = ((b+1)\,!)^k\frac{a}{b} - \sum_{n=0}^{b+1}\frac{((b+1)\,!)^k}{(n\,!)^k} \tag{0.3}$$

It is clear that the right hand side is a positive integer. We now proceed to show that the left hand side is not an integer, thereby getting the desired contradiction. For $n \geq b+2$, we write $n = (b+2)+j$ for $j \geq 0$ and hence

$$\frac{((b+1)\,!)^k}{(n\,!)^k} = \left(\frac{(b+1)\,!}{((b+2)+j)\,!}\right)^k = \left(\frac{1}{(b+2+j)\cdots(b+2)}\right)^k \leq \left(\frac{1}{(b+2)^{j+1}}\right)^k$$

As a consequence we see that

$$\begin{aligned}\sum_{n=b+2}^{\infty}\frac{((b+1)\,!)^k}{(n\,!)^k} &= \sum_{j=0}^{\infty}\frac{((b+1)\,!)^k}{((b+2)+j)\,!)^k}\\ &\leq \sum_{j=0}^{\infty}\left(\frac{1}{(b+2)^k}\right)^{j+1}\\ &= \frac{1}{(b+2)^k-1}\end{aligned}$$

Note that

$$\frac{1}{(b+2)^k-1} < 1$$

showing that the left-hand side sum in (0.3) can't be an integer. $\square$

(7) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n f(x)\,dx = \frac{1}{2}f(0)$$

Solution:

Note that

$$\int_0^1 nx(1-x^2)^n\,dx = -\frac{n}{2(n+1)}(1-x^2)^{n+1}\Big|_0^1 = \frac{n}{2(n+1)} \tag{0.4}$$

As a result of this, the claim will follow once we show that

$$\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n[f(x)-f(0)]\,dx = 0$$

To this end, let $\epsilon > 0$ be given. Since f is continuous at $x = 0$, there is $1 > \delta_\epsilon > 0$ such that

$$|f(x)-f(0)| < \epsilon, \quad \text{whenever } 0 \le x < \delta_\epsilon$$

First let us note that

$$\begin{aligned}\int_0^1 nx(1-x^2)^n|f(x)-f(0)|\,dx &= \int_0^\delta nx(1-x^2)^n|f(x)-f(0)|\,dx \\ &\quad + \int_\delta^1 nx(1-x^2)^n|f(x)-f(0)|\,dx \\ &= I_n + II_n \qquad (0.5)\end{aligned}$$

We estimate each of the summands I_n and II_n. For I_n, we use the continuity of f at 0. Thus

$$\begin{aligned}I_n = \int_0^\delta nx(1-x^2)^n|f(x)-f(0)|\,dx &\le \epsilon\int_0^1 nx(1-x^2)^n\,dx \\ &= \frac{\epsilon n}{2(n+1)} \\ &< \epsilon\end{aligned}$$

Next we estimate II_n. For this, we use the boundedness of f on $[0,1]$. Suppose $|f(x)| \le M$ for some $M > 0$ and all $0 \le x \le 1$. Let us notice that for sufficiently large n, the function $g(x) = n(1-x^2)^n$ satisfies

$$0 \le g(x) \le n(1-\delta^2)^n \quad \text{whenever} 0 \le \delta \le 1$$

In fact this is true provided that $n \ge (\delta^{-2}-1)/2$. Therefore, for such large n, we estimate

$$\begin{aligned}II_n &= \int_\delta^1 nx(1-x^2)^n|f(x)-f(0)|\,dx \\ &\le 2M\int_\delta^1 nx(1-x^2)^n\,dx \\ &\le 2Mn(1-\delta^2)^n\end{aligned}$$

Therefore, from (0.5) and the above estimates we find that for sufficiently large n

$$\int_0^1 nx(1-x^2)|f(x)-f(0)|\,dx = I_n + II_n \le \epsilon + 2Mn(1-\delta^2)^n$$

On noting that

$$\lim_{n\to\infty} n(1-\delta^2)^n = 0$$

we find that

$$0 \leq \limsup_{n\to\infty} \int_0^1 nx(1-x^2)|f(x)-f(0)|\,dx \leq \epsilon$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\limsup_{n\to\infty} \int_0^1 nx(1-x^2)|f(x)-f(0)|\,dx = 0$$

Therefore, we have

$$\lim_{n\to\infty} \int_0^1 nx(1-x^2)|f(x)-f(0)|\,dx = 0 \tag{0.6}$$

as claimed. Finally, using (0.4) and (0.6) we find that

$$\begin{aligned}\lim_{n\to\infty} \int_0^1 nx(1-x^2)^n f(x) &= \lim_{n\to\infty} \int_0^1 nx(1-x^2)^n [f(x)-f(0)]\,dx \\ &\quad + \lim_{n\to\infty} f(1) \int_0^1 nx(1-x^2)^n\,dx \\ &= \frac{1}{2} f(0)\end{aligned}$$

□

Remark: On the test papers at the 2012 ICMC, the problem was stated incorrectly as $\ldots \lim_{n\to\infty} \int_0^1 nx(1-x^2)^n f(x)\,dx = \frac{1}{2} f(1)$.

(8) Recall that a function $f(x,y)$ is said to be harmonic in an open subset $\mathcal{O}$ of the plane if it is twice continuously differentiable in $\mathcal{O}$ and $f_{xx}(x,y)+f_{yy}(x,y) = 0$ for all (x,y) in $\mathcal{O}$. Let $\mathcal{R}$ be the region in the plane given by

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : x^2 + (y+1)^2 \leq 9 \quad \text{and} \quad x^2 + (y-1)^2 \geq 1\}$$

Show that if f is harmonic in an open disk containing $\mathcal{R}$, then

$$\iint_{\mathcal{R}} f(x,y)\,dxdy = 9\pi f(0,-1) - \pi f(0,1)$$

Solution:

First we rewrite the given double integral as

$$\iint_{\mathcal{R}} f(x,y)\,dx\,dy = \iint_{\mathcal{D}_1} f(x,y)\,dx\,dy - \iint_{\mathcal{D}_2} f(x,y)\,dx\,dy \tag{0.7}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the closed disks

$$\mathcal{D}_1 = \{(x,y) : x^2 + (y+1) \le 9\} \qquad \text{and} \qquad \mathcal{D}_2 = \{(x,y) : x^2 + (y-1)^2 \le 1\}$$

The following property of a function $f(x,y)$ that is harmonic in an open set containing a disk $\mathcal{D}$ centered at (x_0, y_0) can be interpreted as a "Mean Value Property":

$$\frac{1}{\text{area}\,(\mathcal{D})} \iint_{\mathcal{D}} f(x,y)\,dx\,dy = f(x_0, y_0). \tag{0.8}$$

We now apply formula (0.8) to (0.7) to get

$$\begin{aligned}
\iint_{\mathcal{R}} f(x,y)\,dx\,dy &= \iint_{\mathcal{D}_1} f(x,y)\,dx\,dy - \iint_{\mathcal{D}_2} f(x,y)\,dx\,dy \\
&= \text{area}(\mathcal{D}_1) f(0,-1) - \text{area}(\mathcal{D}_2) f(0,1) \\
&= 9\pi f(0,-1) - \pi f(0,1).
\end{aligned}$$

To prove the Mean Value Property (0.8), let r be the radius of $\mathcal{D}$, and consider any disk $\mathcal{E}$ of radius $0 < \rho \le r$ centered at (x_0, y_0). We give the curve $\partial\mathcal{E}$ a counterclockwise orientation, and apply Green's Theorem to find

$$\begin{aligned}
0 &= \iint_{\mathcal{E}} [f_{xx}(x,y) + f_{yy}(x,y)]\,dx\,dy = \iint_{\mathcal{E}} [(f_x)_x(x,y) - (-f_y)_y(x,y)]\,dx\,dy \\
&= \int_{\partial\mathcal{E}} [-f_y(x,y)\,dx + f_x(x,y)\,dy] \\
&= \int_0^{2\pi} (\rho f_y(x_0 + \rho\cos\theta, y_0 + \rho\sin\theta)\sin\theta + \rho f_x(x_0 + \rho\cos\theta, y_0 + \rho\sin\theta)\cos\theta)\,d\theta
\end{aligned}$$

In the last integral we used the following parametrization for the counterclockwise oriented circle $\partial\mathcal{E}$.

$$x = x_0 + \rho\cos\theta, \qquad y = y_0 + \rho\sin\theta, \qquad 0 \le \theta \le 2\pi$$

On dividing both sides of the last equation by ρ, we see that for any $0 < \rho \le r$

$$\int_0^{2\pi} (f_y(x_0 + \rho\cos\theta, y_0 + \rho\sin\theta)\sin\theta + f_x(x_0 + \rho\cos\theta, y_0 + \rho\sin\theta)\cos\theta)\,d\theta = 0 \tag{0.9}$$

As a consequence of the relation (0.9), we will show that the following function is a constant.

$$\psi(\rho) := \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + \rho\cos\theta,\, y_0 + \rho\sin\theta)\,d\theta, \qquad 0 < \rho \le r \tag{0.10}$$

In fact, on differentiating ψ, we see that for $0 < \rho < r$

$$\psi'(\rho) = \frac{1}{2\pi} \int_0^{2\pi} [f_x(x_0 + \cos\theta,\, y_0 + \rho\sin\theta)\cos\theta + f_y(x_0 + \rho\cos\theta,\, y_0 + \rho\sin\theta)\sin\theta]\,d\theta$$

Therefore, by (0.9) we see that $\psi'(\rho) = 0$ for all $0 < \rho < r$. Thus, ψ is a constant on $[0, r]$, and

$$\psi(\rho) = \psi(0) = f(x_0, y_0) \quad \text{for} \quad 0 < \rho \leq r$$

In other words, for $0 < \rho \leq r$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} f(x_0 + \rho\cos\theta,\, y_0 + \rho\sin\theta)\, d\theta = f(x_0, y_0)$$

We multiply both sides by $0 < \rho \leq r$, and integrate the resulting equation on $[0, r]$ to get

$$\frac{1}{2\pi} \int_0^r \int_0^{2\pi} \rho f(x_0 + \rho\cos\theta,\, y_0 + \rho\sin\theta)\, d\theta\, d\rho = f(x_0, t_0) \int_0^r \rho\, d\rho$$

Therefore, we obtain

$$f(x_0, y_0) = \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r \rho f(x_0{+}\rho\cos\theta,\, y_0{+}\rho\sin\theta)\, d\rho\, d\theta = \frac{1}{\text{area}(\mathcal{D})} \iint_{\mathcal{D}} f(x, y)\, dx\, dy$$

This proves the claim (0.8). $\square$

Remark: On the test papers at the 2012 ICMC, the last sentence of the problem was stated as ... *Show that if f is harmonic in an open set containing $\mathcal{R}$,* ... which is an insufficient hypothesis. In particular, such an open set might not be large enough to include $(0, 1)$ in the domain of f.