1. It is not possible to achieve any order of elimination by carefully choosing an elimination parameter if n is even. Consider the following order of elimination: $1, n, n-1, n-2, \ldots, 3$. If the first person to be eliminated is person 1, then the parameter must be of the form nk, for some integer k; therefore the elimination parameter must be even. If the the last person to be eliminated is person 3 (with person 2 surviving), the parameter must be odd. The elimination parameter can't be both even and odd.

2. If x represents the length of the cut, let $P(x)$, $A(x)$ and $V(x)$ represent the perimeter of the base, the area of the base, and the volume of the resulting box as a function of x. Observe that:

$$A(x) = \frac{\sqrt{3}}{4}(1-2\sqrt{3}x)^2$$

$$P(x) = 3(1-2\sqrt{3}x)$$

$$V(x) = \frac{\sqrt{3}}{4}x(1-2\sqrt{3}x)^2$$

The result is implied by maximizing $V(x)$ using the tools of calculus.

3. The function

$$f(x,y) = \frac{x^{n+1}y}{x^{2(n+1)}+y^2}$$

is a function for which

$$\lim_{x\to 0} f(x, g_P(x)) = 0$$

Let $y = x^{n+1}$; then

$$\lim_{x\to 0} f(x,y) \neq 0$$

4. If n and b are relatively prime, then there is a minimal non zero r for which $b^r \equiv 1 \mod n$, or $b^r - 1 = kn$, for some k. It follows that

$$k = d_{r-1}b^{r-1} + \cdots + d_1 b + d_0 \text{ for } 0 \le d_i < b$$

Putting the last two ideas together, we obtain:

$$\frac{1}{n} = \frac{d_{r-1}b^{r-1} + \cdots + d_1 b + d_0}{b^r}\frac{1}{1-b^{-r}}$$

Since $\frac{1}{1-b^{-r}} = \sum_{i=0}^{\infty}(b^{-r})^i$, a cycle length for $\frac{1}{n}$ is r. To show that there is no lower cycle length, note that the equation (which would follow if m were another cycle length)

$$\frac{1}{n} = \frac{e_{m-1}b^{m-1} + \cdots + e_1 b + e_0}{b^m} \frac{1}{1 - b^{-m}}$$

implies that $m \geq r$ because of the minimality of r.

5. Imagine that the three pegs are occupied, from left to right, by the red stack and then the blue stack, and the rightmost peg is unoccupied. Label these three pegs P_L, P_M, and P_R (for left peg, middle peg, and right peg). In order to swap the positions of the two stacks, we will create a double stack containing $2(n-1)$ disks of alternating color (the pattern will be, from the top; red, blue, red, blue, and so on) over the largest blue disk on P_M. We will then be able to move: 1. the largest red disk to P_R; 2. the double stack over the largest red disk on P_R; 3. the largest blue disk onto P_L; 4. the double stack onto the largest blue disk on P_L; 5. the largest red disk onto P_M. Finally, we unstack the double stack. Let $D_M(k)$ denote the number of moves required to create a double stack on P_M. Let $A(k)$ denote the number of moves required to implement the entire algorithm. Since it will require $2(2^k - 1)$ moves to shift the double stack from one peg to another, we would have the following:

$$A(n) = 2D_M(n-1) + 3 + 4(2^{n-1} - 1)$$

Suppose that one has two stacks of k red and blue disks. Let $D_R(k)$ denote the number of moves required to shuffle these two stacks into a single double stack which sits atop P_R. In order to create a double stack of $2k$ disks over the P_M, one is required to create a double stack of $2(k-1)$ disks over P_R. Once this has been done, the second largest red disk would be moved atop the second largest blue disk on P_M, and then the double stack of $2(k-1)$ disks would be moved from P_R onto P_M. We would then have:

$$D_M(k) = D_R(k-1) + 2(2^{k-2} - 1) + 1$$

On the other hand, in order to create a double stack of $2k$ disks on P_R, we are required to create a double stack of $2(k-1)$ disks on P_M, move a red disk from P_L peg onto P_R, move the double stack of $2(k-1)$ disks onto the leftmost peg, move a blue disk from the center peg onto the rightmost peg, the move the double stack onto the rightmost peg. So:

$$D_R(k) = D_M(k-1) + 2 + 4(2^{k-1} - 1)$$

Putting these two together, we would have the relation:

$$D_M(k) - D_M(k-2) = 3 + 2(2^{k-1} - 1) + 4(2^{k-2} - 1)$$

Since $D_M(0) = 0$ and $D_M(1) = 1$, this relation permits an efficient calculation of $D_M(k)$ for any k by examining the telescoping sum (for $i = 2$ or $i = 3$):

$$(D_M(k) - D_M(k-2)) + (D_M(k-2) - D_M(k-4)) + \cdots + (D_M(i) - D_M(i-2)) = D_M(k) - D_M(i-2)$$

6. Assume that $x_n \neq f(x_{n-1})$. Otherwise, a fixed point clearly exists and the limit converges to this fixed point. In particular, $x_1 \neq f(x_1)$. The mean value theorem implies that

$$|x_n - x_m| < \frac{|x_1 - f(x_1)|}{2^{m-3}}$$

So, for any $\epsilon > 0$, choose N so that $\frac{|x_1 - f(x_1)|}{2^{N-3}} < \epsilon$. If $n \geq m \geq N$, it follows that

$$|x_n - x_m| < \frac{1}{2^{m-3}} |x_1 - f(x_1)|$$

This implies that the sequence is Cauchy, therefore convergent. Since the limit exists, it must be that

$$f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n$$

7. The following function $f : \mathbb{R} \to \mathbb{R} - \{0\}$ is a one to one correspondence:

$$f(x) = \begin{cases} x & \text{if } x \text{ is not a whole number} \\ x + 1 & \text{if } x \text{ is a whole number} \end{cases}$$

8. The matrix in question can be thought of as a polynomial in the variable X with complex coefficients, where X is the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

To be more precise, if A is the matrix described in the problem, then

$$A = I + nX + (n-1)X^2 + \cdots + 2X^{n-1}$$

It can be shown that if v is an eigenvector for X, then v is an eigenvector for A, and that the list of eigenvectors for X is a complete list of eigenvectors for A. If v is an eigenvector for X, then it has the form

$$\begin{pmatrix} 1 \\ \omega_i \\ \omega_i^2 \\ \cdot \\ \cdot \\ \cdot \\ \omega_i^{n-1} \end{pmatrix}$$

where ω_i is an nth root of unity.