1. The solution is $\binom{p+q}{q}$. In order to see this, think of $p+q$ boxes. Each box will contain a card. The deck is divided into two parts, one containing p cards and one containing q cards. To shuffle the two parts together is the same as selecting q of the $p+q$ boxes then placing the cards in order into those boxes.

2. First, recall Wilson's Theorem:

$$(p-1)! \equiv -1 \text{mod} p$$

Proof. Note that for every $0 < i \leq k$, $p - i \equiv -i \bmod p$. As a result

$$(p-(k+1)!k! \equiv (-1)^k (p-(k+1))!(p-k)\cdots(p-1) \bmod p$$

$\square$

The result follows from Wilson's Theorem.

3. Player 1 should take 3, 1, or 2 chips at any stage of the game, depending upon whether or not the number of chips left is equivalent to 0, 2, or 3 modulo 4.

Proof. If a player faces a table which contains $4k+1$ chips and must act, then this player will ultimately lose. The reason for this is that the other player can always arrange that a total of four chips leave the table every round. This will leave the player that faces $4k+1$ chips initially with 1 chip in the final round. The proof is induction on k. It is clear that the choices of player 1 will guarantee this situation for player 2. $\square$

4. Recall the "rank + nullity theorem." The dimension of $\mathbb{R}^n$ is n.

rank $(\cdot)$ + nullity $(\cdot) = n$

where $\cdot$ could be equal to A or B.

Proof. Let the null space of the matrices A and B be spanned by $\{a_1, \ldots, a_p\}$ and $\{b_1, \ldots, b_q\}$ (with $p+q=n$), respectively. The fact that $A+B=I$ implies that the collection of vectors $\{a_1, \ldots, a_p, b_1, \ldots, b_q\}$ will form a basis for $\mathbb{R}^n$. One can then write any vector as a linear combination of the basis vectors and check that the required conditions hold. $\square$

5. The quotient is not defined at $x = 0$. The proof will proceed by noting the Taylor series expansions of the various functions and composing these appropriately.

Proof. The Taylor series for the relevant functions are

$$\sin x = x - \frac{x^3}{3!} + \text{Terms of at least order 5}$$

$$\tan x = x + \frac{x^3}{3} + \text{Terms of at least order 5}$$

$$\arcsin x = x + \frac{x^3}{3!} + \text{Terms of at least order 5}$$

$$\arctan x = x - \frac{x^3}{3} + \text{Terms of at least order 5}$$

After composition and some manipulation, we obtain the following limit

$$\lim_{x \to 0} \frac{-x^3 + \text{Terms of at least order 5}}{x^3 + \text{Terms of at least order 5}}$$

As a consequence, the limit is -1.

$\square$

6. The thing to observe is that we are looking for ordered pairs (x, y) in the first quadrant that satisfy the inequality $\frac{1}{2} < \frac{y}{x} < 3$ whose sum is odd and bounded above by 5. There are only three ordered pairs which satisfy these conditions: $(1,2), (2,3)$, and $(3,2)$. $U+D+R+L = 5$, where the letters represent the number of moves up (U), down (D), and so forth. Let the ordered quadruple (U, D, R, L) represent the set of directions. To get to $(2,3)$, $U = 3$ and $R = 2$. The associated probability is $\frac{5!}{2!3!}(\frac{1}{16})^2(\frac{1}{4})^3$. Similarly, the probability associated to the point $(3,2)$ is $\frac{5!}{3!2!}(\frac{1}{16})^3(\frac{1}{4})^2$. Both $(2,0,2,1)$ and $(3,1,1,0)$ will reach $(1,2)$ The probability associated with the first quadruple is $\frac{5!}{2!2!1!}(\frac{1}{4})^2(\frac{1}{16})^2(\frac{9}{16})$. The probability associated to the second quadruple is $\frac{5!}{3!1!1!}(\frac{1}{4})^3(\frac{1}{8})(\frac{1}{16})$. The sum of these four probabilities is the solution.

7. The difficulty is in finding a way to show that as one gets close to x, the denominator gets large in our rational number. We may as well assume that $x \in [0,1]$ and that x is irrational.

Proof. Let $\epsilon > 0$. There is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. For every $1 \leq i \leq n$, define

$$\delta_i = \min\{|x - 0|, |x - \frac{1}{i}|, \ldots, |x - \frac{i-1}{i}|, |x - 1|\}$$

Define $\delta = \frac{\min \delta_i}{2}$

If $y \in (x - \delta, x + \delta)$ and $y \notin \mathbb{Q}$, then $|f(x) - f(y)| = 0$. If $y \in \mathbb{Q}$, then $y = \frac{p}{q}$, then $q > n$. (If not, then $|x - \frac{p}{q}| < \delta < \delta_q \leq |x - \frac{p}{q}|$) This is sufficient to prove the proposition.

$\square$