1. Two points P and Q are randomly selected in the interval $[0,2]$. What is the probability that they are within $\frac{1}{3}$ of each other?

The area between the lines $P = Q + \frac{1}{3}$ and $P = Q - \frac{1}{3}$ that is contained in the square $[0,2] \times [0,2]$ is equal to $4 - (\frac{5}{3})^2 = \frac{11}{9}$. Divide this number by 4 to get the required probability.

2. Show that the Maclaurin series of

$$f(x) = \frac{x}{1-x-x^2}$$

is equal to $\sum_{n=1}^{\infty} f_n x^n$, where $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$.

Proof. Consider the equation

$$\frac{x}{1-x-x^2} = \sum_{n=1}^{\infty} f_n x^n$$

Multiply both sides of this equation by the $1 - x - x^2$, and expand the right hand side of the equation. The result, after reindexing, is the equation

$$x = f_1 x + (f_2 - f_1)x^2 + \sum_{n=3}^{\infty}(f_n - f_{n-1} - f_{n-2})x^n$$

The result follows by comparing coefficients on both sides of this equation. □

3. $\sigma^3 = (1\ 2\ 3\ 5\ 7\ 8\ 4\ 6)$. We can calculate σ by taking the cube of σ^3. So $\sigma = (1\ 5\ 4\ 2\ 7\ 6\ 3\ 8\ )$

4. Prove that

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1$$

Proof. Let $v = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$. $v \in \mathbb{R}$, and v satisfies the equation $x^3 + 3x - 4 = 0$. Since $x^3 + 3x - 4 = (x-1)(x^2 + x + 4)$ and the discriminant of $x^2 + x + 4$ is -15, the only real solution of $x^3 + 3x - 4 = 0$ is 1. So, $v = 1$. □

5. Show that $n | (n-1)!$ if $n \neq 4$ and n is a composite number.

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. One line of argument is to produce for each i at least α_i factors of $(n-1)!$ which are divisible by p_i. Division may be used to do this. Note that if $x \geq 2$ and $\alpha \geq 2$:

$$x^\alpha - 1 = (x^{\alpha-1} - 1)x + (x - 1)$$

If $p_i = 2$ and $\alpha_i \geq 3$ or if $p_i \neq 2$ and $\alpha_i \geq 2$, $p_i^{\alpha_i - 1} - 1 \geq \alpha_i$. This implies that $(n-1)!$ has at least α_i factors which are divisible by p_i in these cases. The other cases are checked individually.

□

6. Let $f : (a, b) \to \mathbb{R}$. Suppose that f is increasing and satisfies the property that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Prove that f is continuous.

Proof. The condition to which f is subject implies that for $r < s < t$,

$$\frac{f(s) - f(r)}{s - r} \leq \frac{f(t) - f(r)}{t - r} \leq \frac{f(t) - f(s)}{t - s}$$

Now let $\epsilon > 0$. Let $x_0 \in (s, t) \subset (a, b)$. Choose $w \in \mathbb{N}$ large enough so that $(x_0 - \frac{\epsilon}{w}, x_0 + \frac{\epsilon}{w}) \subset (s, t)$. Let $m = \frac{f(t) - f(x_0)}{t - x_0}$. Let k be equal to the larger of w or m. Finally, let $\delta = \frac{\epsilon}{k}$. If $|x - x_0| < \delta$, the inequality above implies that $|f(x) - f(x_0)| < \epsilon$.

□

7. Let $\mathbb{Z}_{\geq a} = \{x \in \mathbb{Z} | x \geq a\}$. Write down a formula for a 1-1 correspondence:

$$\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 1}$$

Proof. Define the set T_n as follows.

$$T_n = \{(p, q, r) | p + q + r = n; p, q, r \in Z_{\geq 0}\}$$

For each j, with $0 \leq j \leq n$, define the set $E_{n,j}$ as follows.

$$E_{n,j} = \{(p, q, j) | p + q + j = n; p, q, j \in \mathbb{Z}_{\geq 0}\}$$

Suppose that $(p, q, r) \in (\mathbb{Z}_{\geq 0})^{\times 3}$. Then

$$F(p, q, r) = \sum_{i=0}^{p+q+r-1} |T_i| + \sum_{r+1 \leq i \leq n} |E_{p+q+r,i}| + (q + 1)$$

In this formula, if $p + q + r = 0$, then $\sum_{i=0}^{p+q+r-1} |T_i|$ will be taken to be 0. In addition, if $r = n$, then $\sum_{r+1 \leq i \leq n} |E_{p+q+r,i}|$ will also be taken to be 0.

□