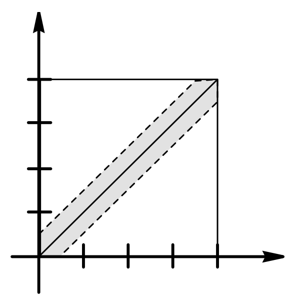
ICMC 2008 - Solutions

1. Let S be a set with a binary operation $*$ that is associative. Suppose that for all x and y in S we have $x * x * x = x$ (i.e. $x^3 = x$) and $x * x * y = y * x * x$ (i.e. $x^2y = yx^2$). Show that for all x and y in S we have that $x * y = y * x$.

   $yx = (yx)^3 = (yx)^2(yx) = ((yx)^2y)x) = y(yx)^2x = y(yx)(yx)x = xy^2yx^2 = xyx^2 = xx^2y = x^3y = xy$

2. Two friends agree to meet at the library, but each has forgotten the time they were supposed to meet. Each remembers that they were supposed to meet sometime between 1:00 pm and 5:00 pm. They each independently decide to go to the library at a random time between 1:00 pm and 5:00 pm, wait for 30 minutes, and leave if the other doesn't show up. What is the probability that they meet during this 4-hour period?

   Solution: Consider the set $[0,4] \times [0,4]$. Draw in the line $y = x$, which represents both students meeting each other at the exact same time. If x arrives at time t, then y can arrive at any time between $t - 0.5$ and $t + 0.5$, and the students will meet. Similarly, if y arrives at time t', then x can arrive at any time between $t' - 0,5$ and $t' + 0.5$, and the students will meet. So, widening the line $y = x$ so that it is everywhere one hour wide and one hour tall forms a region that represents all possible "successful" meeting times. The area of the 4×4-square represents all possible selections of times by both students.

   The area of the shaded region is $16 - 2\left(\frac{1}{2}\right)\left(\frac{7}{2}\right)^2 = \frac{15}{4}$. So, the probability the two students meet during this 4-hour period is $\frac{15/4}{16} = \frac{15}{64}$

3. Suppose that two triangles have a common angle. Show that the sum of the sines of the angles will be larger in that triangle where the difference of the remaining two angles is smaller. HINT: $2\sin\left(\dfrac{\theta+\delta}{2}\right)\cos\left(\dfrac{\theta-\delta}{2}\right) = \sin\theta + \sin\delta$.

   Let the angles of the triangles be denoted by α, β, γ and α', β', γ' and let $\alpha = \alpha'$. We ask under what conditions the following inequality holds: $\sin\alpha + \sin\beta + \sin\gamma < \sin\alpha' + \sin\beta' + \sin\gamma'$, or since $\sin\alpha = \sin\alpha'$, $\sin\beta + \sin\gamma < \sin\beta' + \sin\gamma'$. By the hint, this inequality can be written as $2\sin(\frac{\beta+\gamma}{2})\cos(\frac{\beta-\gamma}{2}) < 2\sin(\frac{\beta'+\gamma'}{2})\cos(\frac{\beta'-\gamma'}{2})$. Since $\alpha = \alpha'$, we have $\beta + \gamma = \beta' + \gamma' < 180°$, so that $2\sin(\frac{\beta+\gamma}{2}) = 2\sin(\frac{\beta'+\gamma'}{2}) > 0$. Dividing both sides by $2\sin(\frac{\beta+\gamma}{2})$ yields $\cos(\frac{\beta-\gamma}{2}) < \cos(\frac{\beta'-\gamma'}{2})$ Since $\cos\theta = \cos|\theta|$ and $y = \cos\theta$ is decreasing on $0° \le \theta \le 180°$, the above inequality holds if and only if $\left|\frac{\beta'-\gamma'}{2}\right| < \left|\frac{\beta-\gamma}{2}\right|$, or equivalently if $|\beta' - \gamma'| < |\beta - \gamma|$.

4. The base of a solid object is the region bounded by the parabola $y = \dfrac{1}{2}x^2$ and the line $y = 2$; cross sections of the object perpendicular to the y-axis are semicircles. What is the volume of the object?

   Cross-sectional area is $\frac{1}{2}\pi\left(\sqrt{2y_i}\right)^2 = \pi y_i$ at level y_i. Volume of one puffed-up cross-section is $\pi y_i \Delta y_i$, and we sum these together to get an approximate volume of $\sum \pi y_i \Delta y_i$. This quickly leads us to the exact volume given by $\int_0^2 \pi y\, dy = 2\pi$.

5. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by $x_0 = 0$, $x_1 = 1$, $x_n = \dfrac{x_{n-1} + (n-1)\, x_{n-2}}{n}$. Determine $\lim\limits_{n\to\infty} x_n$. Recalling that $\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$, $x \in (-1, 1]$ may be helpful.

First, we notice that $x_n = \frac{x_{n-1}+(n-1)x_{n-2}}{n} = \frac{nx_{n-1}-(n-1)x_{n-1}+(n-1)x_{n-2}}{n} = x_{n-1} + \frac{-(n-1)x_{n-1}+(n-1)x_{n-2}}{n}$, or $x_n - x_{n-1} = -\frac{n-1}{n}(x_{n-1} - x_{n-2})$. Similar arugments show $x_{n-1} - x_{n-2} = -\frac{n-2}{n-1}(x_{n-2} - x_{n-3})$, $x_{n-2} - x_{n-3} = -\frac{n-3}{n-2}(x_{n-3} - x_{n-4})$, $x_2 - x_1 = -\frac{1}{2}(x_1 - x_0)$ Substituting, we see $x_n - x_{n-1} = -\frac{n-1}{n} \cdot -\frac{n-2}{n-1} \cdot -\frac{n-3}{n-2} \cdot \cdots \cdot -\frac{1}{2}(x_1 - x_0) = \frac{(-1)^{n-1}}{n}$, $n \geq 2$ We also notice that the above equation is true when $n = 1$.

Since $x_0 = 0$, we can write $x_n = x_n - x_0 = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0) = \sum_{k=1}^{n}(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \sum_{k=0}^{n} \frac{(-1)^k}{k+1}$. Hence $\lim_{n\to\infty} x_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \ln 2$.

6. Prove that $\dfrac{1}{n+1}\dbinom{2n}{n}$ is an integer for all integers $n \geq 1$.

$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1} \cdot \frac{(2n)!}{(n!)^2} = \frac{(2n)!}{(n+1)!n!} = \frac{(2n)!(2n+1-2n)}{(n+1)!n!} = \frac{(2n)!(2n+1)}{(n+1)!n!} - \frac{(2n)!(2n)}{(n+1)!n!} = \frac{(2n+1)!}{(n+1)!n!} - \frac{2n}{n} \cdot \frac{(2n)!}{(n+1)!(n-1)!} = \frac{(2n+1)!}{(n+1)!((2n+1)-(n+1))!} - 2 \cdot \frac{(2n)!}{(n+1)!(2n-(n+1))!} = \binom{2n+1}{n+1} - 2\binom{2n}{n+1}$.

As both $\binom{2n+1}{n+1}$ and $\binom{2n}{n+1}$ are integers for all integers $n \geq 1$, we have $\dfrac{1}{n+1}\dbinom{2n}{n}$ is an integer for all integers $n \geq 1$ as well.

7. Find matrices B and C such that $B^3 + C^3 = \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix}$. Then the characteristic equation of A is $\lambda^2 - 6\lambda + 5 = 0$. By the Cayley-Hamilton Theorem, we have $A^2 - 6A + 5I = 0$. Multiplying both sides by A, we obtain $A^3 - 6A^2 + 5A = 0$. Now consider $(A - 2I)^3$. We have $(A-2I)^3 = A^3 - 6A^2 + 12A - 8I = A^3 - 6A^2 + 5A + 7A - 8I = 7A - 8I$. So, we have $7A = (A-2I)^3 + 8I$, or $A = \left(\frac{1}{\sqrt[3]{7}}(A - 2I)\right)^3 + \left(\frac{2}{\sqrt[3]{7}}I\right)$.

Thus, let $B = \frac{1}{\sqrt[3]{7}}(A - 2I) = \begin{bmatrix} -\frac{1}{\sqrt[3]{7}} & -\frac{1}{\sqrt[3]{7}} \\ 0 & \frac{3}{\sqrt[3]{7}} \end{bmatrix}$ and $C = \frac{2}{\sqrt[3]{7}}I = \begin{bmatrix} \frac{2}{\sqrt[3]{7}} & 0 \\ 0 & \frac{2}{\sqrt[3]{7}} \end{bmatrix}$.