2007 ICMC Solutions

1. Let p and q be distinct primes. Find a polynomial with integer coefficients that has $\sqrt{p}+\sqrt{q}$ as a root.

   Set $x=\sqrt{p}+\sqrt{q}$. Then $x-\sqrt{p}=\sqrt{q}$. Squaring both sides we obtain $x^2-2x\sqrt{p}+p=q$, or $-2x\sqrt{p}=q-p-x^2=-x^2+(q-p)$. Square both sides again to get $4px^2=x^4-2(q-p)x^2+(q-p)^2$. So, $f(x)=x^4-2(q+p)x^2+(q-p)^2$ will suffice.

2. What is the value of the positive integer n for which the least common multiple of 36 and n is 500 greater than the greatest common divisor of 36 and n?

   We know that $\gcd(36,n)=x$ and $\operatorname{lcm}(36,n)=500+x$. Now, since x is a divisor of 36, we have the following possibilities for x: $1,2,3,4,6,9,12,18,36$. Now, 36 must also divide $500+x$; by experimentation we get $x=4$. This tells us that $n=4b$ and that $4b$ must divide 504. Hence, b must divide 126. However, $n=4b$ cannot have 3 as a factor; since $126=2\cdot 3^2\cdot 7$, the possible values of b are 1, 2, 7, and 14. We quickly see 14 is the only possibility. Hence, $n=4\cdot 2\cdot 7=56$.

3. Evaluate: $\lim\limits_{x\to\infty}(x+2)\cdot\int_x^{3x}\frac{dt}{t\sqrt{t^4+1}}$.

   For $x\le t\le 3x$, we have $0\le\frac{1}{t\sqrt{t^4+1}}\le\frac{1}{x\sqrt{x^4+1}}$. Hence, $0<\int_x^{3x}\frac{dt}{t\sqrt{t^4+1}}\le\int_x^{3x}\frac{dt}{x\sqrt{x^4+1}}=\frac{1}{x\sqrt{x^4+1}}\cdot\int_x^{3x}dt=\frac{1}{x\sqrt{x^4+1}}\cdot(2x)=\frac{2}{\sqrt{x^4+1}}$. Thus, $0\le\lim\limits_{x\to\infty}(x+2)\cdot\int_x^{3x}\frac{dt}{t\sqrt{t^4+1}}\le\lim\limits_{x\to\infty}(x+2)\cdot\frac{2}{\sqrt{x^4+1}}=0$. So, $\lim\limits_{x\to\infty}(x+2)\cdot\int_x^{3x}\frac{dt}{t\sqrt{t^4+1}}=0$.

4. Answer the following.

   (a) Let p be a fixed prime. Suppose an integer a is selected at random. What is the probability that a is divisible by p? (Think about the possible remainders when dividing by p.)
   Reduce the integers modulo p, obtaining a uniform distribution over the set $\{0,1,...,p-1\}$. The probability a is divisible by p is the same as the probability of selecting 0 from $\{0,1,...,p-1\}$, which is $\frac{1}{p}$.

   (b) Let p be a fixed prime. Suppose two integers a and b are selected at random. What is the probability that a and b are both divisible by p?
   Selecting a and b are independent events. So, using part (a), we see the probability is $\frac{1}{p}\cdot\frac{1}{p}=\frac{1}{p^2}$.

   (c) Suppose two integers a and b are selected at random. Show that the probability that a and b are relatively prime is $\prod_{p\in P}\left(1-\frac{1}{p^2}\right)$, where P is the set of all primes.
   For each prime p, the probability a and b both have p as a factor is $\frac{1}{p^2}$; hence, the probability a and b are not both divisible by p is $1-\frac{1}{p^2}$. Now, if p_1 and p_2 are distinct primes, whether or not p_1 divides both a and b is independent from whether or not p_2 divides a and b. Hence, the probability that neither p_1 nor p_2 divide both a and b is $\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)$. Continuing, we see that the probability that a and b have no prime factor in common (and hence are relatively prime) is $\prod_{p\in P}\left(1-\frac{1}{p^2}\right)$, where P is the set of all primes.

5. Let A be an $n\times n$ matrix such that $a_{ij}=1$ when $i\neq j$, and $a_{ij}=0$ when $i=j$. In other words, $A=\begin{bmatrix}0&1&\cdots&1\\1&0&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&0\end{bmatrix}$.

   Find A^{-1}. (Using the matrix $B=\begin{bmatrix}1&1&\cdots&1\\1&1&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&1\end{bmatrix}$ may be helpful.)

   First, note that $A=B-I$, where I is the $n\times n$ identity matrix, and that $B^2=nB$. For any real number r, we see $(B-I)(rB-I)=rB^2-(r+1)B+I=(rn-(r+1))B+I$. So, $rB-I$ will be the inverse of $B-I$ if $rn-(r+1)=0$, or $r=\frac{1}{n-1}$. Hence, $A^{-1}=\frac{1}{n-1}B-I=\begin{bmatrix}\frac{2-n}{n-1}&\frac{1}{n-1}&\cdots&\frac{1}{n-1}\\\frac{1}{n-1}&\frac{2-n}{n-1}&\cdots&\frac{1}{n-1}\\\vdots&\vdots&\ddots&\vdots\\\frac{1}{n-1}&\frac{1}{n-1}&\cdots&\frac{2-n}{n-1}\end{bmatrix}$.

6. Let g and h be noncommuting elements in a group of odd order. If g and h satisfy the relations $g^3=e$ and $ghg^{-1}=h^3$, determine the order of h.

   Note that $h^9=\left(ghg^{-1}\right)^3=gh^3g^{-1}=g\left(ghg^{-1}\right)g^{-1}=g^2hg^{-2}$. So, $h^{27}=\left(ghg^{-1}\right)^9=gh^9g^{-1}=g\left(ghg^{-1}\right)^3g^{-1}=g\left(g^2hg^{-1}\right)g^{-1}=g^3hg^{-3}$. Since $g^3=e$, $h^{27}=h$, or $h^{26}=e$. Since the group has odd order, the only possibilities for the order of h are 1 and 13. Since g and h do not commute, $h\neq e$; hence, $|h|=13$.