1. Find the maximum value of the function $f(x)=\sqrt{4x-x^2+12}-\sqrt{10x-x^2-24}$.

   Rewriting, we notice that $f(x)=\sqrt{16-(x-2)^2}-\sqrt{1-(x-5)^2}$. So, the values of $f(x)$ are the differences of the y-coordinates of two semicircles, and these semicircles are internally tangent at the point $(6,0)$. Further, we notice that the domain of $f(x)$ is $[4,6]$. Since the function $g(x)=\sqrt{16-(x-2)^2}$ is decreasing on $(4,6)$, the largest the expression $\sqrt{4x-x^2+12}$ can be on $[4,6]$ occurs at $x=4$, and at this point, $g(4)=\sqrt{4(4)-4^2+12}=2\sqrt{3}$. Further, at $x=4$, the function $h(x)=\sqrt{1-(x-5)^2}$ is as small as it can get; namely, $h(4)=0$. Hence, the maximum value of $f(x)=g(x)-h(x)$ is $f(4)=2\sqrt{3}$.

2. Evaluate $\sum_{n=1}^{\infty}\arctan\left(2/n^2\right)$ .Hint: $\arctan\alpha-\arctan\beta=\arctan\left((\alpha-\beta)/(1+\alpha\beta)\right)$.

   Using the hint, we see $\arctan(1/n)-\arctan(1/(n+2))=\arctan\left(2/(n+1)^2\right)$. Let S_N denote the Nth partial sum of the given series. For $N\geq 3$, $S_N=\sum_{n=1}^{N}\arctan\left(2/n^2\right)=\arctan 2+\sum_{n=1}^{N-1}\arctan\left(\frac{2}{(n+1)^2}\right)=\arctan 2+\sum_{n=1}^{N-1}\left(\arctan\left(\frac{1}{n}\right)-\arctan\left(\frac{1}{n+2}\right)\right)=\arctan 2+\arctan 1+\arctan\left(\frac{1}{2}\right)-\arctan\left(\frac{1}{N}\right)-\arctan\left(\frac{1}{N+1}\right)$. As $N\to\infty$, we have $S_N\to\arctan 2+\arctan 1+\arctan\left(\frac{1}{2}\right)=\frac{3\pi}{4}$. The last equality holds because $\arctan 1=\frac{\pi}{4}$ and, if $x\neq 0$, we have $\arctan x+\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}$.

3. Let X be a continuous random variable having the probability density function $f(x)=\left\{\begin{array}{ll}\frac{1}{x^2}, & x\geq 1\\ 0, & \text{otherwise}\end{array}\right.$.
   Find $P\left(\left[\sqrt{10X}\right]=20\;\middle|\;\left[\sqrt[3]{X}\right]=3\right)$. (Recall that $[x]$ is the greatest integer less than or equal to x.)

   In order for $\left[\sqrt{10X}\right]=20$, we need $20\leq\sqrt{10X}<21$, or $40\leq X<44.1$. Given $\left[\sqrt[3]{X}\right]=3$, we know $3\leq\sqrt[3]{X}<4$, or $27\leq X<64$. So, we find $P\left(\left[\sqrt{10X}\right]=20\;\middle|\;\left[\sqrt[3]{X}\right]=3\right)=\left(\int_{40}^{44.1}\frac{1}{x^2}\,dx\right)\Big/\left(\int_{27}^{64}\frac{1}{x^2}\,dx\right)=\left(\frac{1}{40}-\frac{1}{44.1}\right)/\left(\frac{1}{27}-\frac{1}{64}\right)=\frac{984}{9065}$.

4. A bicyclist rides 18 miles in exactly 72 minutes. Prove that there exists a contiguous 3-mile segment within this 18 miles that the rider completed in exactly 12 minutes.

   For $0\leq x\leq 15$, let $T(x)$ be the time in minutes that it takes the rider to go from point x to point $x+3$. Certainly, $T(x)$ is continuous on $[0,15]$. Notice that $T(0)+T(3)+T(6)+T(9)+T(12)+T(15)=72$. It is not possible for all of the values on the left-hand side to be strictly greater than 12, nor is it possible for all of the values on the left-hand side of the equation to be strictly less than 12. Hence, there exist $r,s\in\{0,3,6,9,12,15\}$ with $T(r)\leq 12$ and $T(s)\geq 12$. Since $T(x)$ is continuous, the Intermediate Value Theorem guarantees that there exists a k between r and s so that $T(k)=12$. In other words, the rider travels from point k to point $k+3$ in exactly 12 minutes.

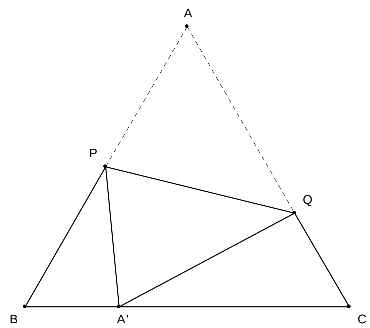
5. Let S be a set of real numbers that is closed under multiplication. Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T and U is closed under multiplication.

   By contradiction, assume that neither T nor U are closed under multiplication. Thus, there exist $t_1,t_2\in T$ such that $t_1t_2\notin T$ and $u_1,u_2\in U$ such that $u_1u_2\notin U$. Since $S=T\cup U$, we have $t_1t_2\in U$ and $u_1u_2\in T$. Consider the element $t_1t_2u_1u_2$. Since the product of any three elements of T (resp. T) is in T (resp. U), we have $t_1t_2u_1u_2=(t_1t_2)u_1u_2\in U$ and $t_1t_2u_1u_2=t_1t_2(u_1u_2)\in T$. So, $t_1t_2u_1u_2\in T\cap U$. However, T and U are disjoint, a contradiction. Thus, either T or U is closed under multiplication

6. Let r and s be nonzero integers. Prove that the equation $\left(r^2-s^2\right)x^2-4rsxy-\left(r^2-s^2\right)y^2=1$ has no solutions in integers x and y.

   Suppose x and y are integers satisfying the given equation. Factoring the left-hand side of the equation, we see $((r-s)x-(r+s)y)((r+s)x+(r-s)y)=1$. Since r,s,x,y are integers, we have the product of two integers, and this product must be 1. Thus, the factors in the product are either both 1 or both -1. Solving the system $(r-s)x-(r+s)y=\delta$, $(r+s)x+(r-s)y=\delta$, where $\delta=\pm 1$, we obtain $x=\frac{r\delta}{r^2+s^2}$ and $y=-\frac{s\delta}{r^2+s^2}$. Squaring each of these and adding, we see $x^2+y^2=1$, meaning $x=\pm 1$ and $y=0$, or $x=0$ and $y=\pm 1$. However, if $x=\pm 1$ and $y=0$, we get from the above system of equations $r-s=\pm 1$ and $r+s=\pm 1$, giving $s=0$. If $x=0$ and $y=\pm 1$, we get $-r-s=\pm 1$ and $r-s=\pm 1$, giving $r=0$. Since r and s were assumed to be nonzero integers, this is a contradiction.

7. Equilateral triangle ABC has been creased and folded along $\overline{PQ}$ so that vertex A lies at the point A' on $\overline{BC}$. If $BA' = 1$ and $A'C = 2$, find the length of $\overline{PQ}$.



Let $x = PA = PA'$ and $y = QA = QA'$. Then $PB = 3 - x$ and $QC = 3 - x$. Applying the Law of Cosines to $\triangle PBA'$, we have $x^2 = 1^2 + (3-x)^2 - 2\,(1)\,(3-x)\cos 60°$. Solving for x, we get $x = \frac{7}{5}$. Applying the Law of Cosines to $\triangle QCA'$, we have $y^2 = 2^2 + (3-y)^2 - 2\,(2)\,(3-y)\cos 60°$. Solving for y, we get $y = \frac{7}{4}$. Finally, applying the Law of Cosines to $\triangle PA'Q$, we have $PQ^2 = \left(\frac{7}{5}\right)^2 + \left(\frac{7}{4}\right)^2 - 2\left(\frac{7}{5}\right)\left(\frac{7}{4}\right)\cos 60°$. Solving for PQ, we get $PQ = \frac{7\sqrt{21}}{20}$.

8. An evil genie pops out of a lamp and presents you with twelve identical-looking stones. The genie tells you that one stone has an imperfection not visible to the naked eye that causes the stone to be either slightly heavier or slightly lighter than the others. The genie also presents you with a balance scale and says you may use this scale at most three times. The genie will give you your heart's desire if you can correctly identify the imperfect stone AND determine whether it is heavier or lighter than the others.

   Label the twelve stones $\{A, B, C, D, E, F, G, H, I, J, K, L\}$. Weighing 1 is $\{A, B, C, D\}$ versus $\{E, F, G, H\}$. If the scales balance, go to 1. If the scales do not balance, go to 2.

   1. The scales balanced, meaning the odd stone is in $\{I, J, K, L\}$.Weighing 2 is $\{I, J, K\}$ versus the good stones $\{A, B, C\}$. If the scales balance, go to 1a. If the scales do not balance, go to 1b.

      1a. The odd stone must be L. Weighing L against any other stone will determine whether L is heavier or lighter than the others.

      1b. The odd stone is in $\{I, J, K\}$, and it is light (resp. heavy) because $\{I, J, K\}$ is lighter (resp. heavier) than $\{A, B, C\}$. Weighing 3 is I versus J. If they balance, then K is the odd stone and light (resp. heavy). If they do not balance, then the light (resp. heavy) side is the odd stone.

   2. The scales tipped. Without loss of generality, assume that $\{A, B, C, D\}$ was the heavier side. So, we either have an odd, heavy stone among $\{A, B, C, D\}$, or an odd, light stone among $\{E, F, G, H\}$. Weighing 2 is $\{A, B, E\}$ versus $\{F, C, D\}$. If they balance, go to 2a. If $\{A, B, E\}$ is heavier than $\{F, C, D\}$, go to 2b. If $\{A, B, E\}$ is lighter than $\{F, C, D\}$, go to 2c.

      2a. The odd stone must be in $\{G, H\}$ and it must be light since $\{G, H\}$ were on the light side of the first measurement. Measure G against H, the light side is the odd, light stone.

      2b. $\{A, B, E\}$ was heavier than $\{F, C, D\}$. Neither C nor D can be an odd, heavy stone, and E cannot be an odd, light stone. Thus, either A or B is the odd, heavy stone, or F is the odd, light stone. Weighing 3 is $\{A, F\}$ against any two normal stones (e.g. $\{G, H\}$). If $\{A, F\}$ is heavier, then A is the odd, heavy stone. If $\{A, F\}$ is lighter, then F is the odd, light stone. If they balance, then B is the odd, heavy stone.

      2c. $\{A, B, E\}$ was lighter than $\{F, C, D\}$. Neither A nor B can be an odd, heavy, while F cannot be odd and light. Thus, either C or D is the odd, heavy stone, or E is the odd, light stone. Weighing 3 is $\{C, E\}$ against any two normal stones (e.g. $\{G, H\}$). If $\{C, E\}$ is heavy, then C is the odd, heavy stone. If $\{C, E\}$ is light, then E is the odd, light stone. If they balance, then D is the odd, heavy stone.