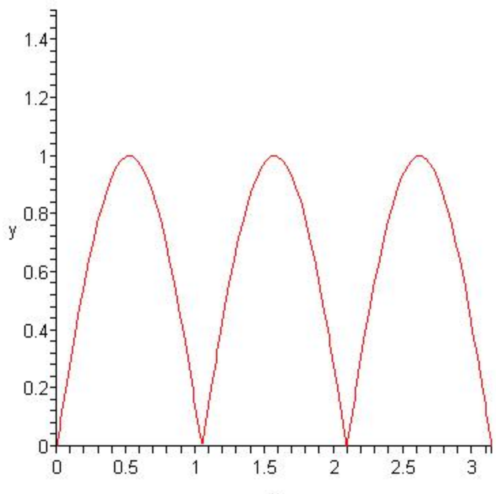
ICMC - IPFW, April 1, 2005

1. Evaluate the integral

$$\int_{-1}^{1} \frac{|\sin(n\cos^{-1} x)|}{\sqrt{1-x^2}} dx .$$

Solution. Since x is in $[-1,1]$ we can make the substitution $\theta = \cos^{-1} x$, with $\theta \in [0, \pi]$. The integral becomes

$$\int_{-1}^{1} \frac{|\sin(n\cos^{-1} x)|}{\sqrt{1-x^2}} dx = -\int_{\pi}^{0} |\sin n\theta|\, d\theta = \int_{0}^{\pi} |\sin n\theta|\, d\theta$$



From the graph of $|\sin n\theta|$ (seen above for n=3) the integral can be calculated as

$$\int_{0}^{\pi} |\sin n\theta|\, d\theta = n \int_{0}^{\pi/n} \sin n\theta d\theta = -n \frac{\cos n\theta}{n} |_{0}^{\pi/n} = 2 .$$

2. All points in the plane are colored in red, white, or blue. Prove that there is at least one pair of points of the same color with distance between them one unit.

Solution. By contradiction, assume that there exists a red-white-blue coloring of the points in the plane such that no two points of the same color are at distance 1 of each other.
We will show that for such a coloring there are no two points of different colors at a distance of $\sqrt{3}$ units. Indeed, suppose such a pair does exist. Without loss of generality we can assume it is a red-white pair, and denote the two points R and W respectively. If we construct a rhombus $RBWA$ with sides 1 and diagonal RW, we see that the other two vertices, A and B must be colored blue. However, it is easy to see that diagonal AB has length 1, which contradicts our original assumption.

Therefore, for the coloring in question all points at a distance $\sqrt{3}$ have the same color. This implies that the circle $x^2 + y^2 = 3$ will have to be colored the same color as the origin, which leads to a contradiction with our assumption that no two points of the same color are at distance 1 of each other (just choose two points on this circle that are distance 1 of each other).

3. Find the limit of the sequence defined by

$$a_n = \frac{1}{n^3}\sum_{k=1}^{n} \ln(1+kn).$$

Justify your answer.

Solution.

$$0 \le a_n = \frac{1}{n^3}\sum_{k=1}^{n}\ln(1+kn) = \frac{1}{n^3}\ln\prod_{k=1}^{n}(1+kn) \le \frac{1}{n^3}\ln\prod_{k=1}^{n}(1+n^2) = \frac{1}{n^3}\ln(1+n^2)^n$$

$$= \frac{1}{n^2}\ln(1+n^2) \le \frac{1}{n^2}\ln(1+n)^2 = \frac{2}{n^2}\ln(1+n) \le \frac{2}{n^2}(1+n) = \frac{2n+2}{n^2}.$$

Thus, $0 \le a_n \le \dfrac{2n+2}{n^2}$, so using the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$.

4. Let $\mathbb{Q}^+$ be the set of all positive rational numbers, and let "*" be an operation on $\mathbb{Q}^+$ that satisfies the following identities for all $a,b,c,d \in \mathbb{Q}^+$:

$$\begin{aligned}&(a*b)\cdot(c*d) = (a\cdot c)*(b\cdot d),\\ &a*a = 1,\\ &a*1 = a,\end{aligned}$$

where the operation "·" is the usual multiplication of numbers in $\mathbb{Q}^+$.

Compute the value of the expression $((6/5)*(8/15))*2$.

Solution 1. If we let $b = a$ and $d = 1/a$ in the first equation we obtain

$$(a*a)(c*(1/a)) = (ac)*1 = ac,$$

that is

$$c*(1/a) = ca.$$

If we use now $1/a$ instead of a we obtain $c*a = c/a$, so operation "*" is just the usual division of rational numbers. Therefore

$$((6/5)*(8/15))*2 = ((6/5)\div(8/15))\div 2 = (6/5)(15/8)(1/2) = 9/8.$$

Solution 2.

Lemma: x*y=x ÷y.

Proof:
1. $y\cdot(1*y)=(y*1)\cdot(1*y)=(y\cdot 1)*(1\cdot y)=y*y=1$, so $1*y$ is the reciprocal of y (by uniqueness of inverses in $\mathbb{Q}^+$).
2. $x*y=(x\cdot 1)*(1\cdot y)=(x*1)\cdot(1*y)=x\cdot y^{-1}$.
From the Lemma:
$((6/5)*(8/15))*2=((6/5)\div(8/15))\div 2=((6/5)(15/8))(1/2)=9/8$.

5. Let I be the identity matrix in the set $M_n(\mathbb{R})$ of real n by n square matrices. Consider matrices A and B in $M_n(\mathbb{R})$ such that $A^3=A^2$ and $A+B=I$. Show that the matrix $AB+I$ is nonsingular and find its inverse.

Solution. Since $B=I-A$

$AB+I=A(I-A)+I=I+A-A^2$, and $I-A+A^2$ is an inverse:

$(I+A-A^2)(I-A+A^2)=(I-A+A^2)+A(I-A+A^2)-A^2(I-A+A^2)=I-A+A^2+A-A^2+A^3-A^2+A^3-A^4$
$=I-A^2+A^3+A^3-A^4$.

The last four terms cancel to I because $A^4=A^3A=A^2A=A^3=A^2$.

One should also cite the linear algebra fact that in the ring $M_n(\mathbb{R})$ it is enough to check just one product, or explicitly check that the product in the other order also gives the identity by an analogous computation (then, this solution works for any ring).

6. Determine the real constants a, b, c, and p, such that

$$\lim_{x\to\infty}\left[\sqrt{9x^4-24x^3+6x^2+5}-(ax^p+bx+c)\right]=\frac{7}{3}.$$

Solution. We have

$$\sqrt{9x^4-24x^3+6x^2+5}-(ax^p+bx+c)=x^2\left[\sqrt{9-\frac{24}{x}+\frac{6}{x^2}+\frac{5}{x^4}}-x^{p-2}\left(a+\frac{b}{x^{p-1}}+\frac{c}{x^p}\right)\right].$$

If $p\neq 2$ or if $p=2$ and $a\neq 3$ then the limit is $\pm\infty$, so $p=2$ and $a=3$. Substituting these values and rationalizing the numerator the limit becomes

$$\lim_{x\to\infty}\frac{9x^4-24x^3+6x^2+5-(3x^2+bx+c)^2}{\sqrt{9x^4-24x^3+6x^2+5}+(3x^2+bx+c)}=\lim_{x\to\infty}\frac{(-24-6b)x^3+(6-b^2-6c)x^2+\ldots}{x^2\left(\sqrt{9-\frac{24}{x}+\frac{6}{x^2}+\frac{5}{x^4}}+3+\frac{b}{x}+\frac{c}{x^2}\right)}.$$

For the limit to be equal to $7/3$ we must have $-24-6b=0$ and $\frac{6-b^2-6c}{6}=\frac{7}{3}$, which gives $b=-4$ and $c=-4$.

7. Let M and N be the midpoints of BC and CD in the parallelogram ABCD, and let P be the intersection of AM and BN. Determine the ratios $\frac{AP}{AM}$ and $\frac{BP}{BN}$.

Solution. Extend AM until it crosses DC in D' and extend BN until it crosses AD in A'. Triangle ABP is similar to D'NP (since D'N is parallel to AB).
Then

$$\frac{BP}{NP}=\frac{AB}{ND'}=\frac{AB}{NC+CD'}=\frac{AB}{\frac{1}{2}AB+AB}=\frac{2}{3},$$

since CD'=AB, because of the fact that triangles MCD' and MBA are congruent (A.S.A.)
But from this equation we get

$$\frac{BP}{NP+BP}=\frac{2}{3+2}\text{ , i.e. }\frac{BP}{BN}=\frac{2}{5}.$$

Similarly, since triangles BPM and PAA' are similar, we have

$$\frac{MP}{AP}=\frac{BP}{PA'}=\frac{MB}{AA'}=\frac{MB}{2AD}=\frac{\frac{1}{2}AD}{2AD}=\frac{1}{4}.$$ (Here we used the fact that AD=DA', since triangles A'DN and BNC are congruent).

Therefore

$$\frac{MP+AP}{AP}=\frac{1+4}{4}\text{, i.e. }\frac{AP}{AM}=\frac{4}{5}.$$

8. Given the integers $x,\ y,$ and $z,$ prove that if 25 divides the sum $x^5+y^5+z^5,$ then 25 divides at least one of the numbers $x^5+y^5,\ x^5+z^5,$ or y^5+z^5.

Solution. We write $x=5k+x_1,\ y=5l+y_1$, and $z=5m+z_1$, where k,l,m,x_1,y_1,z_1 are integers, and without loss of generality we can assume that $0\le x_1\le y_1\le z_1\le 5$.
If 25 divides $x^5+y^5+z^5,$ using the Binomial Theorem we see that 25 must also divide $x_1^5+y_1^5+z_1^5$. We will show that $x_1=0$, which implies that 25 divides y^5+z^5.
For any integer a, we have $a^5\equiv a(\mathrm{mod}\,5),$ and since 5 divides $x_1^5+y_1^5+z_1^5$, then 5 must divide $x_1+y_1+z_1$.
By contradiction, suppose $x_1>0$. Then there are only four choices for the triple $(x_1,y_1,z_1),$ namely,

$$(1,1,3),\ (1,2,2),\ (2,4,4),\text{ and }(3,3,4),$$

since 5 divides $x_1+y_1+z_1$ and since $0\le x_1\le y_1\le z_1\le 5$.
We have
$1^5\equiv 1(\mathrm{mod}\,25),\ 2^5\equiv 7(\mathrm{mod}\,25),\ 3^5\equiv 18(\mathrm{mod}\,25),$ and $4^5\equiv -1(\mathrm{mod}\,25).$
By inspection, for none of the four choices above the sum $x_1^5+y_1^5+z_1^5$ is divisible by 25, which represents a contradiction. Therefore, $x_1=0$ and the conclusion follows.