ICMC - April 2nd, 2004 - Solutions

1. Partition the set $\{1, 2, 3, 4, 5\}$ into two arbitrarily chosen sets. Prove that one of the sets contains two numbers and their difference.

   We attempt to partition $\{1, 2, 3, 4, 5\}$ into two sets A and B in such a way that neither set contains two numbers *and* their difference. Thus, 2 cannot be in the same set as either 1 or 4, else we would have $2-1=1$ or $4-2=2$. So, put 2 in A, and put 1 and 4 in B. If we put 3 in B, then we have $4-3=1$. So, 3 must go in A. Similarly, placing 5 in B leads to $5-4=1$; thus, 5 cannot be in B. However, 5 cannot be in A since $5-3=2$. We have reached a contradiction. Hence, no matter how the two sets are constructed, one of the two sets must contain two numbers and their difference.

2. Suppose $a > 1$.

   (a) Show the series $\sum_{n=0}^{\infty} \frac{2^n}{a^{2^n}+1}$ converges.
   Since $\lim_{n\to\infty} \left| \left(\frac{2^{n+1}}{a^{2^{n+1}}+1}\right) \Big/ \left(\frac{2^n}{a^{2^n}+1}\right) \right| = \lim_{n\to\infty} \left(\frac{2^{n+1}}{2^n} \cdot \frac{a^{2^n}+1}{a^{2^{n+1}}+1}\right) = 2\lim_{n\to\infty} \left[\left(1+\frac{1}{a^{2^n}}\right) \Big/ \left(a^{2^n}+\frac{1}{a^{2^n}}\right)\right] = 2\,(0) = 0$, this series converges by the Ratio Test.

   (b) Determine to what value this series converges.

   Since

$$\begin{aligned}
\frac{2^n}{a^{2^n}+1} &= \frac{2^n\left(a^{2^n}-1\right)}{\left(a^{2^n}+1\right)\left(a^{2^n}-1\right)} \\
&= \frac{2^n\left(a^{2^n}-1\right)+2^{n+1}-2^{n+1}}{\left(a^{2^n}+1\right)\left(a^{2^n}-1\right)} \\
&= \frac{2^n\left(a^{2^n}+1\right)-2^{n+1}}{\left(a^{2^n}+1\right)\left(a^{2^n}-1\right)} \\
&= \frac{2^n}{a^{2^n}-1} - \frac{2^{n+1}}{a^{2^{n+1}}-1}
\end{aligned}$$

   the sum $\sum_{n=0}^{\infty} \frac{2^n}{a^{2^n}+1} = \sum_{n=0}^{\infty} \left(\frac{2^n}{a^{2^n}-1} - \frac{2^{n+1}}{a^{2^{n+1}}-1}\right)$ is telescoping. Hence $\sum_{n=0}^{\infty} \frac{2^n}{a^{2^n}+1} = \frac{2^0}{a^{2^0}-1} = \frac{1}{a-1}$.

3. Let A be a 4×4 matrix such that each entry of A is either 2 or -1. Let $d = \det(A)$; clearly, d is an integer. Show that d is divisible by 27.

   Let B be the matrix obtained from A by subtracting row one of A from each of the other three rows. Then $\det A = \det B$. Each entry in the last three rows of B is $-3, 0$, or 3, and therefore divisible by 3. Now let C be the matrix obtained from B by dividing each of the entries in the last three rows of B by 3. All of the entries of C are integers, giving $\det C$ is an integer; moreover, $\det A = \det B = 3^3 \det C$. So, $\det A$ is divisible by 27.

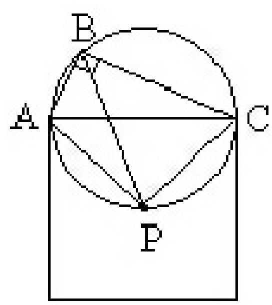
4. Let $a_1, a_2, ..., a_n$ be a finite sequence of real numbers. Form a sequence of length $n-1$ be average two consecutive terms of the sequence: $\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}, ..., \frac{a_{n-2}+a_{n-1}}{2}, \frac{a_{n-1}+a_n}{2}$. Continue this process of averaging two consecutive terms until you have only one term left. Show that this final term is $\frac{\sum_{i=0}^{n-1}\binom{n-1}{i}a_{i+1}}{2^{n-1}}$.

   By induction on n, the length of the sequence: If $n=1$, then the sequence is $a_1 = \frac{\sum_{i=0}^{1-1}\binom{1-1}{i}a_{i+1}}{2^{1-1}}$. So, assume that for any sequence of length k, with $k \le n$, we will have final term $\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}a_{i+1}}{2^{k-1}}$. For $k = n+1$, after the first step we have the sequence $\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}, \cdots, \frac{a_n+a_{n+1}}{2}$, which is a sequence of length n. Thus, by the inductive hypothesis, the final term will be:

$$\begin{aligned}
\frac{\sum_{i=0}^{n-1}\binom{n-1}{i}\left(\frac{a_{i+1}+a_{i+2}}{2}\right)}{2^{n-1}} &= \frac{\sum_{i=0}^{n-1}\binom{n-1}{i}(a_{i+1}+a_{i+2})}{2^n} \\
&= \frac{\sum_{i=0}^{n-1}\binom{n-1}{i}a_{i+1}+\sum_{i=0}^{n-1}\binom{n-1}{i}a_{i+2}}{2^n} \\
&= \frac{a_1+a_{n+1}+\sum_{i=1}^{n-1}\binom{n-1}{i}a_{i+1}+\sum_{i=0}^{n-2}\binom{n-1}{i}a_{i+2}}{2^n} \\
&= \frac{a_1+a_{n+1}+\sum_{i=0}^{n-2}\binom{n-1}{i+1}a_{i+2}+\sum_{i=0}^{n-2}\binom{n-1}{i}a_{i+2}}{2^n} \\
&= \frac{a_1+a_{n+1}+\sum_{i=0}^{n-2}\left(\binom{n-1}{i+1}+\binom{n-1}{i}\right)a_{i+2}}{2^n} \\
&= \frac{a_1+a_{n+1}+\sum_{i=0}^{n-2}\binom{n}{i+1}a_{i+2}}{2^n} \\
&= \frac{a_1+a_{n+1}+\sum_{i=1}^{n-1}\binom{n}{i}a_{i+1}}{2^n} \\
&= \frac{\sum_{i=0}^{n}\binom{n}{i}a_{i+1}}{2^n}
\end{aligned}$$

5. Let P be the center of a square with side $\overline{AC}$. Let B be a point in the exterior of the square such that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AC}$. Prove: $\overline{BP}$ bisects $\angle ABC$.

   Since P is the center of the square, $\triangle APC$ will also be a right triangle. Construct a circle with diameter $\overline{AC}$; both points B and P will be on this circle. (This is because the circle that circumscribes a right trangle has as its center the midpoint of the hypotenuse.) Since $\overline{AP}$ and $\overline{PC}$ are equal chords of this circle, the arcs AP and PC are equal. Thus $\angle ABP = \angle CBP$.



6. Two ferryboats start at the same instant from opposite sides of a river, travelling across the water on routes at right angles to the shores. Each travels at a constant speed, but one is faster than the other. They pass at a point 720 yards from the nearest shore. Both boats remain at their slips 10 minutes before starting back. On their return trips, they meet 400 yards from the nearest shore. How wide is the river?

   Since both boats remain in their slips for the same amount of time, this information does not enter into the solution of the problem. When the ferryboats meet for the first time, the combined distance the boats have travelled is equal to the width of the river. When the boats reach the opposite shore, the combined distance the boats have travelled equals two widths of the river. When they then meet the second time, the combined distances the boats have travelled is three widths of the river. Since the boats move at a constant speed, it follows that each boat has travelled three times as far as when they first met and had travelled a combined distance of one river-width. The slow boat had travelled 720 yards when the boats first met. Thus, by the second meeting, the slow boat has travelled $3 \times 720 = 2160$ yards. Since this second meeting occurs at the point when the slow boat has moved 400 yards away from the far shore, it follows that the width of the river is given by $2160 - 400 = 1760$ yards.