2022 Indiana Collegiate Mathematics Competition
April 9, 2022
Indiana Wesleyan University

Directions:

- Teams may be up to three persons. You are free to talk through problems or divide work as you wish.
- Submit solutions you wish to be graded, but no scratch work.
- Start a new sheet for each problem. If you need more than one page for a solution, be sure to label each page.
- Please use only one side of each paper.
- Label each page with your team name, problem number, and page number.
- Calculators and other aids are not permitted.
- The exam will last for two hours. When the exam is complete, place solutions to be graded in the envelope. Write your team name and problem number(s) on the outside of the envelope.

1. Consider the four lines $l_1 : x = -3$, $l_2 : x = 1$, $l_3 : y = 2$, and $l_4 : y = -4$. If A is some point in the plane, suppose each of the segments from A to the lines meets perpendicularly at B, C, D, and E respectively. Consider the locus of all points A where

$$|AB||AC| = |AD||AE|.$$

   Find an equation describing this locus, and specify as much as possible the type of plane curve it is.

2. Find two non-zero functions $f(x)$ and $g(x)$ so that $f'(x) \neq 0$, $g'(x) \neq 0$, and

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g'(x).$$

3. Consider the function $f(x) = \dfrac{1}{1 - e^{-1/x}}$.
   (a) (2 points) Assuming $x > 0$, find $f'(x)$.
   (b) (8 points) Compute $\displaystyle\int_0^1 \frac{e^{-1/x}}{x^2(1 - e^{-1/x})^2}\,dx$.

4. Let A be a square matrix, and suppose positive integers m and n exist so that $A^m = I$ and $A^n \neq I$. Find

$$\det(I + A + A^2 + ... + A^{m-1}).$$

5. (a) (4 points) Define $n?$ as the sum of integers from 1 to n. For example, $5? = 1 + 2 + 3 + 4 + 5$. Compute the number of zeros that appear at the end of decimal representation of 2022?.
   (b) (6 points) Define $n!$ as the product of integers from 1 to n. For example, $5! = 1 * 2 * 3 * 4 * 5$. Compute the number of zeros that appear at the end of the decimal representation of 2022!.

6. Can a group be the union of two of its proper subgroups?

7. If g is a function, denote $g \circ g \circ ... \circ g$ (m times) as g^m. Suppose that $g : [0, 1] \to [0, 1]$ is continuous and that there is an m so that for all x, $g^m(x) = x$. Show that in fact $g^2(x) = x$.

8. It is well known that $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ have the same cardinality, and the standard classroom demonstration of this involves a diagonal lines argument. Explicitly give a function between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, and show that it is bijective.