# Spring 2019 Indiana Collegiate Mathematics Competition (ICMC) Exam

Mathematical Association of America – Indiana Section

Compiled by:Paul Fonstad (Franklin College)

Provide counterexamples to each of the following statements:

(1) If both $f(x)$ and $g(x)$ are continuous and monotone on $\mathbb{R}$, then $f(x) + g(x)$ is continuous and monotone on $\mathbb{R}$.

(2) If a function $f(x)$ is not bounded in any neighborhood of a point $a$, then either $\lim\limits_{x \to a^+} |f(x)| = \infty$ or $\lim\limits_{x \to a^-} |f(x)| = \infty$.

(3) A function cannot be continuous at only one point in its domain and discontinuous everywhere else.

(4) If a function is differentiable and increasing on an interval $(a, b)$, then its derivative is positive on the interval $(a, b)$.

(5) If $f(x)$ is a function with an antiderivative $F(x)$ that is defined at both $a$ and $b$, then $\int_a^b f(x)\ dx = F(b) - F(a)$.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $f(x) = \frac{x^2}{4}$, and consider the set of all right triangles in the plane whose right angle vertex lies at the origin and whose other two vertices lie somewhere else on the graph of $y = f(x)$. Conjecture a point other than the origin that all such triangles must pass through, and then prove your conjecture.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Two-Face and his henchmen are once again on a crime spree terrorizing the good people of Gotham City. However, rather than flipping a coin to decide his behavior, Two-Face has decided to adopt a more deterministic approach. When Two-Face's gang crashes a party with 60 guests in attendance, they first line everyone up against the wall. Then, the first henchman walks down the line and takes \$10 from every guest. Next, a second henchman walks down the line giving \$10 to the second, fourth, sixth, etc. party guests. This process is repeated with a third henchman taking \$10 from the third guest and every third guest thereafter, then with a fourth henchman giving \$10 to the fourth guest and every fourth guest thereafter, and so on, until finally the sixtieth henchman gives \$10 to only the sixtieth (last) person in line.

(1) How many party guests made money? How many lost money? How many broke even?

(2) Which party guest(s) made the most money?

(3) Which party guest(s) lost the most money?

(4) How much money did Two-Face and his henchmen make?

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

A fair coin is a coin that will produce a result of either heads (H) or tails (T) when flipped with equal probability.

(1) If you start flipping a fair coin, what is the expected number of flips needed to get your first tail (T)?

(2) If you start flipping a fair coin, what is the expected number of flips needed to achieve your first string of heads followed by tails (HT)?

(3) If you start flipping a fair coin, what is the expected number of flips needed to achieve your first string of heads followed by heads (HH)?

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $A = \begin{pmatrix} 1 & 2019 \\ 0 & 1 \end{pmatrix}$. For **any** $n \in \mathbb{Z}$, conjecture what the four entries of the matrix $A^n$ are, and then prove your conjecture.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $(s_n)_{n=1}^{\infty}$, $(t_n)_{n=1}^{\infty}$, and $(u_n)_{n=1}^{\infty}$ be sequences with the following properties:

i) $(s_n)_{n=1}^{\infty}$ is monotone decreasing
ii) $(t_n)_{n=1}^{\infty}$ is monotone increasing
iii) $s_n \geq u_n \geq t_n$ for every $n \in \mathbb{N}$.

For each of the three sequences $(s_n)_{n=1}^{\infty}$, $(t_n)_{n=1}^{\infty}$, and $(u_n)_{n=1}^{\infty}$, conjecture whether that sequence must converge, must not converge, or if not enough information is given to determine convergence. Then prove your conjectures.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Suppose that $G$ is a group. A subset $S$ of $G$ is called a set of *generators* for $G$ if every element of $G$ can be written as the finite product of elements in $S$ and/or their inverses.

Suppose that $G$ is a group with identity $e$ which has $\{x, y\}$ as a set of generators, where the generators $x$ and $y$ satisfy the following relations:

i) $x^2 = e$
ii) $y^4 = e$
iii) $xyxyxy = e$.

Determine the maximum number of elements that $G$ can contain. Justify your answer.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Suppose that $T$ is the triangular pyramid with vertices at $(0,0,0)$, $(12,0,0)$, $(0,8,0)$, and $(0,0,24)$. What is the maximum volume that a rectangular prism $R$ which has one vertex at $(0,0,0)$ and which is inscribed in $T$ can have?

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