Spring 2018 Intersectional Collegiate Mathematics Competition (ICMC) Exam

Mathematical Association of America – Illinois, Indiana, and Michigan Sections

Compiled by: Daniel Maxin, Zsuzsanna Szaniszlo, Tiffany Kolba (Valparaiso)

Show that

$$\sin(x)\sin(2x)...\sin(nx) \neq 1$$

for every real number x and any positive integer $n \geq 2$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Determine the smallest natural number n such that

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}} \geq 100.$$

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Suppose a_1, a_2,...,a_n are strictly positive real numbers and
$a_1^x + a_2^x + ... + a_n^x \geq n$ for every real number x. Prove that $a_1 a_2 \cdots a_n = 1$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Show that if $x + y + z > 0$ then

$$\det \begin{bmatrix} x & z & y \\ y & x & z \\ z & y & x \end{bmatrix} \geq 0.$$

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Consider the following sequence defined recursively

$$x_1 = \frac{1}{2}, \quad x_{k+1} = x_k^2 + x_k, \quad k \geq 1.$$

Find the integer part of S_{100} where

$$S_{100} = \frac{1}{x_1 + 1} + \frac{1}{x_2 + 1} + ... + \frac{1}{x_{100} + 1}.$$

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Consider a semicircle and AB its diameter. Pick two arbitrary points D and E on the semicircle such that the segments (AD) and (BE) intersect at M in the interior of the semicircle. Prove that

$$|AM| \cdot |AD| + |BM| \cdot |BE| = |AB|^2.$$

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Suppose a fair six-sided die is rolled and let X denote the outcome of the die roll. Suppose a fair coin is flipped until X heads are obtained. Compute the expected value of the number of flips.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

A town has n inhabitants who like to form clubs. They want to form clubs so that every pair of clubs should share a member, but no three clubs should share a member. What is the maximum number of clubs they can form? Illustrate with an example.

Show work to be graded below, and use the reverse side of the page to continue if necessary.