Spring 2017
Indiana Collegiate Mathematics Competition (ICMC)
Exam

Mathematical Association of America – Indiana Section

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Directions: There are a total of 8 problems on this exam. Before beginning the exam, please record your team number from the front of this envelope on the top of each problem sheet in the space provided. Blank sheets of scratch paper are provided, but make sure that all work that you wish to have graded is recorded in the indicated space on each problem sheet. You may use calculators when completing the exam, but you may not access the internet or use any other electronic resources. At the end of the exam session, place all problem sheets and the provided straight edge and compass back in the envelope.

PROBLEM 1 Team Number:

In constructible geometry, one constructs points, lines, and circles from given points, lines, and circles, using an unmarked straight edge and compass.

- The straight edge draws a line between points already given, which includes the line segment connecting them; the line may extend as far beyond either point as desired. New points are created where the line intersects already existing lines or circles.
- The compass draws a circle (or a circular arc) centered on a given point with a radius extending to another point from the center. Again, new points are created where the circle intersects other circles or lines.

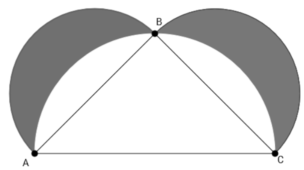
(a) Use the provided straight edge and compass to construct the midpoint M of the line segment $\overline{AB}$ given below.

Show your construction below. Label M in your construction. Do not erase any intermediate steps of your construction.

(b) Use the provided straight edge and compass to construct a square $ABCD$ with the line segment $\overline{AB}$ given below as one of its sides. Do not erase any intermediate steps in your construction.

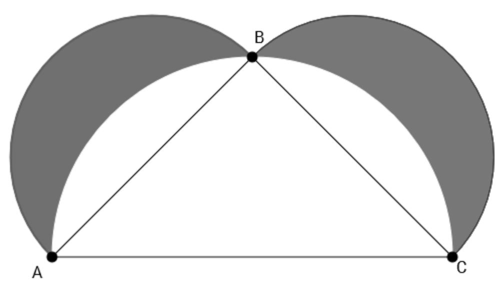
Show your construction below. Label C and D in your construction. Do not erase any intermediate steps of your construction.

(c) Given an isosceles right triangle ABC, one can use a compass to construct lunes as follows. First, one semicircle is formed with the line segment $\overline{AC}$ as its diameter. Two other semicircles are formed with $\overline{AB}$ and $\overline{BC}$ as their diameters. The lunes are the shaded shapes in the figure below.



Use the provided straight edge and compass to construct a square whose area is equal to the area of **one** of the lunes on the diagram given below. Justify how you know the square you construct has the appropriate area.

Show work to be graded below, and use the reverse side of the page to continue if necessary. Do not erase any intermediate steps of your construction.



Consider the following series

$$\sum_{n=1}^{\infty} (a_n)^n$$

where

$$a_n = \begin{cases} \frac{1}{3^1} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} & \text{if } n \text{ is odd.} \\ |\sin n \cos n| & \text{if } n \text{ is even.} \end{cases}$$

Make a conjecture as to whether the series converges or not, and then prove your conjecture.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Two students, Joe and Frank, are each asked to independently select a number at random from the interval [0,1] in such a way that each number in [0,1] is just as likely to be chosen as any other number in this interval. If a denotes the number chosen by Joe and b denotes the number chosen by Frank, what is the probability that the quadratic equation $x^2 + ax + b = 0$ has at least one real root?

Show work to be graded below, and use the reverse side of the page to continue if necessary.

A *multiplicative magic square* is an $n \times n$ square array of numbers consisting of n^2 distinct positive integers (not necessarily consecutive) arranged such that the product of the n numbers in any of the n rows, n columns, or 2 main diagonal lines is always the same number. Call this common product the *magic product.*

(a) Show that the magic product of a 3×3 multiplicative magic square must be a perfect cube.

(b) Find an example of a 3×3 multiplicative magic square whose magic product is minimal. Explain how you know this magic product is minimal.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

For any positive integer n, let $s(n)$ be the sum of the first n terms of the sequence

$$0, 1, 1, 2, 2, 3, 3, 4, 4, \ldots, k, k, k+1, k+1, \ldots$$

(a) Find a formula for $s(n)$. (Note: Your final formula should not have "$\ldots$" in it.)

(b) Suppose that m and n are any two positive integers with $m > n$. Prove that $s(m+n) - s(m-n) = mn$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Suppose that A is an $n \times n$ matrix such that every entry of A is ± 1. Show that the determinant of A is divisible by 2^{n-1}.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

A robot is programmed to shuffle cards in such a way so that it always rearranges cards in the same way relative to the order in which the cards are given to it. The thirteen hearts arranged in the order

$$A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$$

are given to the robot, shuffled, and then the shuffled cards are given back to the robot and shuffled again. This process is repeated until the cards have been shuffled a total of 7 times. If at this point the order of the cards is

$$2, 4, 6, 8, 10, Q, A, K, J, 9, 7, 5, 3,$$

what was the order of the cards after the first shuffle?

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Suppose A is a non-empty, closed[1] subset of $\mathbb{R}$ such that for each $a \in A$, every open interval that contains a also contains another element of A. Show that A must be uncountable.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

[1] A is closed in $\mathbb{R}$ if and only if $\mathbb{R} - A$ consists of the union of any number of open intervals.