# Spring 2016 Indiana Collegiate Mathematics Competition (ICMC) Exam

Mathematical Association of America – Indiana Section

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Assume we are looking at numbers in normal decimal representations, and consider the set $A = \{1, 11, 111, 1111, \ldots\} = \{x \in \mathbb{Z} \,|\, x \text{ consists entirely of 1s}\}$. For all $a \in A$, define $n(a)$ to be the number of 1s in $a$'s representation. (For example, $n(1) = 1$ and $n(111) = 3$.) Prove or disprove the following statement: The set $\{a \in A \,|\, n(a) \text{ divides } a\}$ is infinite.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function with the following properties:

(i) $f(2) = 2$

(ii) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$

(iii) $f(m) > f(n)$ whenever $m > n$

Conjecture what all the possible candidates for $f$ are, and then prove your conjecture.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Arrange 8 points in 3-space so that each of the 56 possible triplets of points determined forms an isosceles triangle. Prove your arrangement works. (Note: Some triangles may be degenerate!)

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $A$ be a subset of $\mathbb{R}$, and let $f, g : A \to A$ be two continuous functions. Then $f$ is said to be homotopic to $g$ if there is a continuous function $h : A \times [0,1] \to A$ such that $h(a,0) = f(a)$ and $h(a,1) = g(a)$ for all $a$ in $A$. (This function is sometimes called a deformation function. It may be helpful to think of $[0,1]$ as being a "slider" that continuously morphs the function $f$ into the function $g$.)

(1) Let $A = [0,1] \subset \mathbb{R}$, and define $f, g : A \to A$ by $f(a) = a^2$ and $g(a) = a^3$. Find a function $h$ demonstrating that $f$ and $g$ are homotopic.

(2) Recall that an equivalence relation $\sim$ on a set $X$ requires three properties:
- *Reflexivity*: $\forall\, x \in X$, $x \sim x$
- *Symmetry*: $\forall\, x, y \in X$, if $x \sim y$, then $y \sim x$
- *Transitivity*: $\forall\, x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$

Prove that for any subset $A$ of $\mathbb{R}$, being homotopic forms an equivalence relation on the set $\mathcal{C}(A)$ of all continuous functions $f : A \to A$.

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The Fibonacci numbers $f_n$ ($n = 0, 1, 2, \ldots$) are defined recursively by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-2} + f_{n-1}$ for $n \geq 2$.

(1) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(f_n, f_{n+1}) = (f_{n+1}, f_{n+2})$ for all $n \geq 0$, and then state and prove a conjecture for what $T^n(0, 1)$ equals for all $n \geq 1$.

(2) Find the matrix $A$ of $T$ with respect to the standard basis for $\mathbb{R}^2$, and then find the eigenvalues of $A$.

(3) Find a non-recursive expression for $f_n$ for all $n \geq 1$.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $A$ be an open subset of the real numbers and $f : A \to \mathbb{R}$ be a differentiable function such that $f'(a) = 0$ for all $a \in A$.

(1) Prove that if $A = \mathbb{R}$, $f$ must be a constant function. (Note: Do not use antiderivatives in your proof, as the proof that antiderivatives are unique up to a constant depends on this statement.)

(2) Prove or disprove: If $A \neq \mathbb{R}$, $f$ must be a constant function.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $G$ be a set and $* : G \times G \to G$ be a binary operation. If $*$ satisfies the following conditions:

- *Associativity*: $\forall\, a, b, c \in G$, $(a * b) * c = a * (b * c)$
- *Identity*: $\exists e \in G$ such that $\forall\, g \in G$, $e * g = g * e = g$
- *Inverse*: $\forall\, a \in G$, $\exists\, a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

we say $G$ is a group. For purposes of shorthand, the symbol $*$ is often omitted (i.e., we write $a * b$ as simply $ab$ and $a * a * a$ as simply $a^3$).

Assume that $G$ is a group which has the following additional properties:

(i) For all $g, h \in G$, $(gh)^3 = g^3 h^3$.
(ii) There are no elements in $G$ of order 3 (i.e., $\nexists\, g \in G$ such that $g^3 = e$).

For any group $G$ with properties (i) and (ii) listed above:

(1) Prove that for all $a, b \in G$, if $a \neq b$, then $a^3 \neq b^3$.
(2) Prove that the map $\phi : G \to G$ defined by $\phi(g) = g^3$ is a bijection if $G$ is finite.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

(**Note: This is a puzzle problem. The length is for clarity, not necessarily its difficulty.**) Assume elections are conducted by voters who place all the candidates in rank order. Under these conditions there are several possible voting methods available:

- Plurality – the candidate with the most first-place votes wins.
- Plurality with Elimination – the following process is repeated until there's a winner:
  - If there's a candidate that has more than half the first-place votes, he/she wins.
  - Otherwise, eliminate the candidate(s) with the fewest first-place votes. Reorder the voters' preferences by moving other candidates up in the ranks to fill the vacancies.
- Borda count – candidates receive points based upon each voter's ranking. A last-place vote earns a candidate one point; a second-to-last-place vote earns a candidate two points; a third-to-last-place vote earns a candidate three points; and so forth, with a first-place vote earning a candidate $n$ points, where $n$ is the total number of candidates. The candidate with the most total points wins.
- Pairwise Comparison – candidates receive points based upon their performance in a round-robin style analysis. Each candidate is paired with each other candidate. (For example, if there were four candidates, there would be six possible matchups.) For each matchup, count the number of voters who prefer one candidate over the other. The winner of the matchup (more voter preferences) receives a point; if there's a tie, both candidates receive half a point. The candidate with the most total points wins.
- Survivor – the following process is repeated until there's a winner:
  - If there's only one candidate remaining, he/she wins.
  - Otherwise, eliminate the candidate(s) with the most last-place votes. Reorder the voters' preferences by moving other candidates up in the ranks to fill the vacancies.

**Example:** Consider the following sample preference table for 9 voters in an election with 3 candidates: $X$, $Y$, and $Z$:

| Rank/Number of Votes | 4 | 3 | 2 |
|---|---|---|---|
| First | $X$ | $Y$ | $Z$ |
| Second | $Y$ | $Z$ | $Y$ |
| Third | $Z$ | $X$ | $X$ |

That is, 4 voters rank $X$ as their first choice, $Y$ as their second choice, and $Z$ as their third choice; 3 voters would pick $Y$ first, $Z$ second and $X$ third; and 2 voters would choose $Z$ first, $Y$ second, and $X$ third. Using this table of preferences the winners for the various voting methods would be:

- Plurality – $X$ wins with the most first-place votes (4 of them)
- Plurality with Elimination – no candidate has more than half the first place votes, so we eliminate the candidate(s) with the fewest. Candidate $Z$ is eliminated. The new preference table is

  | Rank/Number of Votes | 4 | 3 | 2 |
  |---|---|---|---|
  | First | $X$ | $Y$ | $Y$ |
  | Second | $Y$ | $X$ | $X$ |

  Now $Y$ wins with more than half the first-place votes.
- Borda count – the point totals are as follows:
  - $X$: 4(3)+3(1)+2(1)=17
  - $Y$: 4(2)+3(3)+2(2)=21
  - $Z$: 4(1)+3(2)+2(3)=16

  Therefore, $Y$ wins.
- Pairwise Comparison – the matchups are as follows:
  - $X$ vs. $Y$: $Y$ wins 5 to 4 because 5 voters like $Y$ better than $X$; $Y$ gets a point
  - $X$ vs. $Z$: $Z$ wins 5 to 4 because 5 voters like $Z$ better than $X$; $Z$ gets a point
  - $Y$ vs. $Z$: $Y$ wins 7 to 2 because 7 voters like $Y$ better than $Z$; $Y$ gets a point

  Therefore, $Y$ wins.
- Survivor – we eliminate the candidate(s) with the most last-place votes. Candidate $X$ is eliminated. The new preference table is

  | Rank/Number of Votes | 4 | 3 | 2 |
  |---|---|---|---|
  | First | $Y$ | $Z$ | $Y$ |
  | Second | $Z$ | $Y$ | $Z$ |

  Now $Z$ is eliminated, leaving only $Y$ remaining; therefore, $Y$ wins.

**Challenge:** Construct a preference table for an election with five candidates – $A$, $B$, $C$, $D$, and $E$ – such that $A$ wins via plurality, $B$ wins via plurality with elimination, $C$ wins via Borda count, $D$ wins via pairwise comparison, and $E$ wins via survivor. You may use any number of voters you would like.

**Show work to be graded on the reverse side of this sheet.**