Spring 2015 Indiana Collegiate Mathematics Competition (ICMC) Exam

Mathematical Association of America – Indiana Section

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Say that an integer n has a super-3 representation if there is a positive integer m, a sequence of distinct nonnegative integers $p_1, \ldots, p_m$, and a sequence $a_1, \ldots, a_m$ where each a_k is ± 1, so that

$$n = \sum_{k=1}^{m} a_k \cdot 3^{p_k} = a_1 \cdot 3^{p_1} + \cdots + a_m \cdot 3^{p_m}.$$

For instance, the integer 8 has the super-3 representation $8 = 3^2 - 3^0$ and the integer -11 has the super-3 representation $-11 = -3^2 - 3^1 + 3^0$. The number 0 has the *empty* super-3 representation; i.e., where $m = 0$ and the sum has no terms.

(1) Give a super-3 representation of 2015.
(2) Prove that every integer n has a super-3 representation.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Prove that, for every positive integer n,

$$\sum_{k=0}^{n}\sum_{i=0}^{n-k}\binom{n}{k}\binom{n-k}{i}2^{k+i}=5^n.$$

Show work to be graded below, and use the reverse side of the page to continue if necessary.

PROBLEM 3 Team Number:

Three pairwise perpendicular line segments $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ have endpoints all on a sphere of unknown radius, and intersect inside the sphere at a point X. Given the lengths $AX = 1$, $CX = 2$, $EX = 3$, and $BX = 4$, determine, with proof, the volume of the octahedron with vertices A, B, C, D, E, and F.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

You are standing in a room, which we will call Σ, which contains eight light switches numbered 1 through 8, all in the off position. On the other side of the door is another room, Λ, which contains eight lights, labeled A through H. Each switch controls exactly one light. Your goal is to determine which switches control which lights. You do this by making a number of trials, which consist of putting some set of switches in room Σ in the on position, then entering room Λ to discover which lights are on. For instance, one trial might be to turn switches 1 and 3 to the on position (and all others off) and observe which lights in room Λ are on (perhaps lights D and H, though of course you can't know this ahead of time).

(1) Give a strategy for finding out which switches control which lights using the smallest possible number of trials. (For this part, you do not need to prove that the number of trials you use is minimal.)

(2) Prove that your strategy uses the minimal number of trials; that is, prove that there is no strategy that determines which switches control which lights using fewer trials.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called C^∞ if all of its derivatives exist everywhere.

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ function with infinitely many zeros in the interval $[0, 1]$. Show that there is some $x \in [0, 1]$ such that $f^{(n)}(x) = 0$ for every integer $n \geq 0$ (that is, such that f and all its derivatives vanish at x).

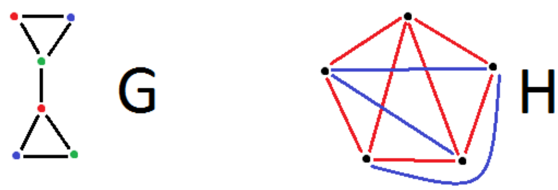
(2) Give an example of such a function f which is nonconstant on every interval. (You need not prove that your function works.)

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Recall that a graph is called *planar* if it can be drawn on the plane in such a way that no two of its edges cross. Further recall that a graph is c-colorable (for a positive integer c) if its vertices can be colored with c colors in such a way that no two adjacent vertices have the same color. The famous *Four Color Theorem* says that every planar graph can be 4-colored.

Call a graph k-color-planar if it can be drawn on the plane, and its *edges* colored with k colors, in such a way that no two edges with the same color cross. Thus, a 1-color-planar graph is just a planar graph.

Shown below are two graphs illustrating these definitions. Graph G is planar and can be 3-colored as shown, and graph H is 2-color planar (but not planar).



Prove that, for every positive integer k, there is a positive integer c_k such that every k-color-planar graph is c_k-colorable. (You may use the Four Color Theorem in your proof if you wish.)

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let S be a finite set and $* : S \to S$ be a binary operation on S. Suppose that $*$ satisfies the following two conditions:

- $*$ is associative; that is, for any $a, b, c \in S$, $(a * b) * c = a * (b * c)$.
- For any $a, b \in S$, $a * (a * (b * a)) = b$.

(1) Prove that $*$ is commutative; that is, that for any $a, b \in S$, $a * b = b * a$.
(2) Prove that $|S|$, the size of S, is a power of 3.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $[0, 1]$, and suppose that $f(0) = f'(0) = 0$ and $f(1) = 1$. Prove that there is some $a \in (0, 1)$ such that $f'(a)f''(a) = \frac{9}{8}$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.