Spring 2014 Indiana Collegiate Mathematics Competition (ICMC) Exam

Mathematical Association of America – Indiana Section

Written by: The Mathematics Faculty of Indiana University - Purdue University Fort Wayne
Edited by: Justin Gash, Franklin College

Let $a > 0$, and define the following function:

$$f(x) = \frac{\sqrt{a^3 x} - a\sqrt[3]{a^2 x}}{a - \sqrt[4]{a x^3}}.$$

- Calculate these limits:

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \\ \lim_{x \to a} f(x) &= \\ \lim_{x \to +\infty} f(x) &= \end{aligned}$$

- Find the maximum value of $f(x)$ on its domain.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let f be a function with domain $(0, \infty)$ satisfying:

- $f(x) = f(x^2)$ for all $x > 0$
- $\lim_{x \to 0^+} f(x) = \lim_{x \to +\infty} f(x) = f(1)$

Show that $f(x)$ is a constant function on $(0, \infty)$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let V be a corner of a right-angled box and let x, y, z be the angles formed by the long diagonal and the face diagonals starting at V. For

$$A = \begin{bmatrix} \sin x & \sin y & \sin z \\ \sin z & \sin x & \sin y \\ \sin y & \sin z & \sin x \end{bmatrix}$$

show that $|\det(A)| \leq 1$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let $f(t)$ be a real valued integrable function on $[0, 1]$, so that both sides of the following equation are continuous functions of x:

$$2x - 1 = \int_0^x f(t)dt.$$

Prove that if $f(t) \leq 1$ for $0 \leq t \leq 1$, then there exists a unique solution $x \in [0, 1]$ of the equation.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let $ABCD$ be a rectangle. The bisector of the angle ACB intersects AB at point M and divides the rectangle $ABCD$ into two regions: the triangle MBC with area s and the convex quadrilateral $MADC$ with area t.

- Determine the dimensions of the rectangle $ABCD$ in terms of s and t.
- If $t = 4s$, what is the ratio AB/BC?

Show work to be graded below, and use the reverse side of the page to continue if necessary.

In a badly overcrowded pre-school, every child is either left-handed or right-handed, either blue-eyed or brown-eyed, and either a boy or a girl. Exactly half of the children are girls, exactly half of the children are left-handed and exactly one fourth of the children are both. There are twenty-six children who are brown-eyed. Nine of those twenty-six are right-handed boys. Two children are right-handed boys with blue eyes. Thirteen children are both left-handed and brown-eyed. Five of these thirteen are girls.

- How many students does the pre-school have?
- How many girls are right-handed and blue-eyed?

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let $n > 1$ be an integer. Let $(G, \cdot)$ be a group, with an identity element e and an element $a \in G$ with $a \neq e$ and $a^n = e$. Let $(H, *)$ be a group, let $f : G \to H$ be an arbitrary function, and then define $F : G \to H$ by:

$$F(x) = f(x) * f(a \cdot x) * f(a^2 \cdot x) * \ldots * f(a^{n-1} \cdot x)$$

- Show that if $f(G)$ is a subset of some Abelian subgroup of H, then F is not a one-to-one function.
- Let $(H, *)$ be the symmetric group $(S_3, \circ)$ (the six-element group of permutations of three objects). Give an example of $(G, \cdot)$, n, and a as above, and a function $f : G \to H$, so that the expression F is a one-to-one function.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Determine whether the following sum of real cube roots is rational or irrational:

$$\sqrt[3]{6+\sqrt{\frac{847}{27}}}+\sqrt[3]{6-\sqrt{\frac{847}{27}}}$$

Show work to be graded below, and use the reverse side of the page to continue if necessary.