Spring 2013 Indiana Collegiate Mathematics Competition (ICMC) Exam

Mathematical Association of America – Indiana Section

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Define a sequence (s_n) recursively as follows: Let $s_1 = 1$ and for $n \geq 1$, let $s_{n+1} = \sqrt{1 + s_n}$. Prove that (s_n) converges, and then find the limit.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let $\mathcal{C}$ be a non-empty collection (possibly infinite) of compact subsets of $\mathbb{R}$.

(1) Prove that $K = \bigcap_{C \in \mathcal{C}} C$ is a compact set.

(2) Give an example that illustrates that the union of a family of compact sets need not be compact.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Assume A and B are two sets with m and n elements, respectively.

(1) How many one-to-one functions are there from A and B?

(2) How many one-to-one and onto functions are there from A to B?

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let p and q be distinct prime numbers. Find the number of generators of the group $\mathbb{Z}_{pq}$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let G be a group and H a subgroup of G with index $(G : H) = 2$. Prove that H is a normal subgroup of G.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

The Fibonacci numbers are defined as

$$f_1 = f_2 = 1$$

and

$$f_{n+1} = f_n + f_{n-1}$$

for $n \geq 3$.

(1) List $f_1, f_2, \ldots, f_7$.
(2) Illustrate, using the list from (a), that $f_{2n+1} = f_{n+1}^2 + f_n^2$ for $n = 1, 2, 3$.
(3) Prove that $f_{2n+1} = f_{n+1}^2 + f_n^2$ for all $n \in \mathbb{N}$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.

Let a, b, m, M be real numbers with $0 < m \leq a \leq b \leq M$, prove that

$$\frac{2\sqrt{mM}}{m+M} \leq \frac{2\sqrt{ab}}{a+b}$$

Show work to be graded below, and use the reverse side of the page to continue if necessary.

A soccer ball is stitched together using white hexagons and black pentagons. Each pentagon borders five hexagons. Each hexagon borders three other hexagons and three pentagons. Each vertex is of valence 3 (meaning that at each corner of a hexagon or pentagon, exactly three hexagons or pentagons meet). How many hexagons and how many pentagons are needed to make a soccer ball? **Hint:** Euler's Polyhedron Formula states that $V - E + F = 2$, where V is the number of vertices, E is the number of edges (i.e., the line adjoining two vertices) and F is the number of faces (hexagons or pentagons).

Show work to be graded below, and use the reverse side of the page to continue if necessary.