# Spring 2012 Indiana Collegiate Mathematics Competition (ICMC) Exam

Mathematical Association of America – Indiana Section

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PROBLEM 1 Team Number:

Show that $n^2$ divides $(n+1)^n - 1$ for any positive integer $n$.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

PROBLEM 2 Team Number: ____________

How many zeros are at the end of $213!$?

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial of degree $n \geq 2$, with integer coefficients, such that $a_0, a_1, a_n$ and $a_2 + \cdots + a_n$ are odd integers. Show that $p(x)$ has no rational root. Give example to show that the conclusion may not be true if any of $a_0, a_1$, $a_n$ or $a_2 + \cdots + a_n$ is even.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $A$ be an $n \times n$ matrix whose diagonal entries are all equal to the same real number $\alpha \in \mathbb{R}$ and all other entries are equal to $\beta \in \mathbb{R}$. Show that $A$ is diagonalizable, and compute the determinant of $A$.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Let $G$ be a group of order 26. If $G$ has a normal subgroup of order 2, show that $G$ is a cyclic group.

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Show that for any positive integer $k$, the following is an irrational number.

$$\sum_{n=0}^{\infty} \frac{1}{(n\,!)^k}$$

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

PROBLEM 7 Team Number:

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 nx(1-x^2)^n f(x)\,dx = \frac{1}{2} f(0)$$

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**

Recall that a function $f(x,y)$ is said to be harmonic in an open subset $\mathcal{O}$ of the plane if it is twice continuously differentiable in $\mathcal{O}$ and $f_{xx}(x,y)+f_{yy}(x,y)=0$ for all $(x,y)$ in $\mathcal{O}$. Let $\mathcal{R}$ be the region in the plane given by

$$\mathcal{R}=\{(x,y)\in\mathbb{R}^2 : x^2+(y+1)^2\le 9 \quad\text{and}\quad x^2+(y-1)^2\ge 1\}.$$

Show that if $f$ is harmonic in an open disk containing $\mathcal{R}$, then

$$\int\!\!\int_{\mathcal{R}} f(x,y)\,dxdy = 9\pi f(0,-1)-\pi f(0,1)$$

**Show work to be graded below, and use the reverse side of the page to continue if necessary.**