PROBLEM 1 TEAM #

Suppose that a group of children are playing a game. An even number of them (say n) are seated in a circle. One of these children is designated as the first child (or child number 1). The child to the right of child number one is designated as child number 2, and so on around the circle in a clockwise fashion.

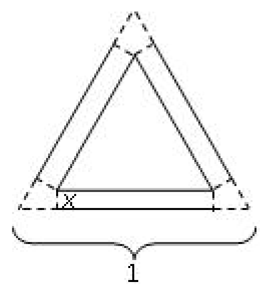
There is a child in the center of the circle. This child is able to choose a number i (which can be larger than n). This number i is known as the elimination parameter. Starting with child number 1, the child in the center of the circle counts off the first i children in a clockwise fashion (moving around the circle more than once if necessary). The child following the i th child is then eliminated from the game. The pattern continues until only one child is left, and that child is declared the winner.

One can make a list of the children as they are eliminated, establishing an order of elimination. Is it possible for the child in the center of the circle to choose an elimination parameter which will realize any elimination order?

Show work to be graded below, and use the reverse side of the page to continue if necessary

PROBLEM 2 TEAM #

Suppose that we have a sheet of cardboard in the shape of an equilateral triangle of side length 1. You cut the corners off of the triangle by making a cut of length x units perpendicular to each side. At that point, the sides are then folded up and the box is formed. Show that the volume of the box is maximized when the area of the base of the box is equal to the total area of the sides of the box.



Show work to be graded below, and use the reverse side of the page to continue if necessary

Let n be a positive integer. For every integer i with $0 \leq i \leq n$, let C be the collection of polynomials defined as follows:

$$C = \{y - x^i - a_1 x^{i-1} - \cdots - a_{i-1} x | a_i \in \mathbb{R}\}$$

Note that if $P \in C$, the equation $P = 0$ defines y as a function of x. For any $P \in C$ let $g_P(x)$ be the function defined by the equation $P = 0$.

Suppose that $f : \mathbb{R}^2 \to R$ is a function such that for every $P \in C$

$$\lim_{x \to 0} f(x, g_P(x)) = 0$$

Either show that

$$\lim_{(x,y) \to (0,0)} f(x, y) = 0$$

or give a counterexample which shows that the above statement is false.

Show work to be graded below, and use the reverse side of the page to continue if necessary

Let b be a positive integer greater than 1. If x is a number, then there is an expansion of x with base b:

$$x = \sum_{i \in \mathbb{Z}} a_i b^i \qquad 0 \leq a_i < b$$

Let n be a positive integer such that $(n, b) = 1$. Show that the period of the expansion of $\frac{1}{n}$ is the smallest positive integer r such that $b^r \equiv 1 \mod n$.

Show work to be graded below, and use the reverse side of the page to continue if necessary

Suppose that one has three pegs and two different stacks of n disks. Each stack of disks sits on a different peg (so there is one peg with no disks on it). The disks in one stack are painted red, and the disks in the other stack are painted blue. Each disk in the red stack has a different radius. The disk at the top of the red stack has the smallest radius, the second topmost disk has the second smallest radius, and so on all the way down to the bottom of the stack (which will be the disk with the largest radius). For each disk in the red stack, there is a disk in the blue stack identical to it except for color. The disks in the blue stack are arranged in exactly the same way as the disks in the red stack.

The goal is to exchange the stacks of disks. There are some rules which limit possible moves. One can only move one disk at a time, and one cannot stack a larger disk upon a smaller disk. One is allowed to stack two disks which have the same size, but are different colors.

Devise an algorithm which will exhange the position of the two stacks, and calculate the number of moves that you algorithm requires to perform the exchange.

Show work to be graded below, and use the reverse side of the page to continue if necessary

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that $|f'(x)| \leq \frac{1}{2}$ for every x. Call x a fixed point for $f(x)$ if $f(x) = x$. Let x_1 be an arbitrary real number. Suppose that we define a sequence as follows. Let $x_2 = f(x_1)$; let $x_3 = f(x_2)$, and in general, let $x_n = f(x_{n-1})$. Show that the following limit exists:

$$\lim_{n\to\infty} x_n$$

Also, show that the limit is a fixed point for the function $f(x)$.

Show work to be graded below, and use the reverse side of the page to continue if necessary

PROBLEM 7 TEAM #

Let A and B be two sets. A one to one correspondence is a function $f : A \to B$ which is both one to one and onto. Show that there is a one to one correspondence between $\mathbb{R}$ and $\mathbb{R} - \{0\}$.

Show work to be graded below, and use the reverse side of the page to continue if necessary

Let A be an $n \times n$ matrix with complex entries. A vector $v \in \mathbb{C}^n$ is an eigenvector for A if $Av = \lambda v$ for some $\lambda \in \mathbb{C}$. Characterize the eigenvectors of the following $n \times n$ matrix:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & 1 & 2 & \cdots & n-1 \\ n-1 & n & 1 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 3 & 4 & \cdots & 1 \end{pmatrix}$$

Show work to be graded below, and use the reverse side of the page to continue if necessary