

Suppose that you are holding a deck of n cards. You divide the deck into two stacks. One stack contains p cards and the other stack contains q cards, with $p + q = n$. You shuffle the stacks in the usual way. How many rearrangements of the cards are possible, if p and q are fixed?

Show work to be graded below, and use the reverse side of the page to continue if necessary

PROBLEM 2

TEAM #

Suppose that p is an odd prime and that $0 \leq k \leq p - 1$. Prove that

$$(p - (k + 1))!k! \equiv (-1)^{k+1} \pmod{p}$$

Show work to be graded below, and use the reverse side of the page to continue if necessary

PROBLEM 3

TEAM #

Consider the following game. There are two players, player 1 and player 2. There is a pile of coins, each identical, on the gaming table. Player 1 acts first, and must remove either 1, 2, or 3 coins. Player 2 acts next, and must remove either 1, 2, or 3 coins. The players continue taking turns in the manner described until there are no coins left on the gaming table. The player who selects the last coin is the loser. It is known that player 1 has a strategy that will guarantee a win if the number of chips on the table is equivalent to 0, 2, or 3 modulo 4. Explain what this strategy is, and prove that the strategy will guarantee the win for player 1.

Show work to be graded below, and use the reverse side of the page to continue if necessary

PROBLEM 4

TEAM #

Suppose that A and B are $n \times n$ matrices with real entries with the following two properties:

$$\text{rank}(A) + \text{rank}(B) = n$$

and

$$A + B = I$$

where I is the $n \times n$ identity matrix.

Prove that $A^2 = A$, $B^2 = B$ and $AB = BA$.

Show work to be graded below, and use the reverse side of the page to continue if necessary

PROBLEM 5

TEAM #

Evaluate the following limit:

$$\lim_{x \rightarrow 0} \frac{\sin \arctan x - \tan \arcsin x}{\arcsin \tan x - \arctan \sin x}$$

Show work to be graded below, and use the reverse side of the page to continue if necessary

Consider the lattice $\mathbb{Z} \times \mathbb{Z}$. Imagine that there is a point at the origin. The point will move in one of four directions in the lattice: up (the y coordinate is increased by 1 unit), right (the x coordinate is increased by one unit), left (the x coordinate is reduced by one unit), or down (the y coordinate is reduced by one unit). The probability that the point will move up is $\frac{1}{4}$, the probability that it will move down is $\frac{1}{8}$, the probability that it will move right is $\frac{1}{16}$, and the probability that the point will move left is $\frac{9}{16}$. Suppose that we keep track of the movement of the point for 5 rounds. What is the probability that it winds up in the open sector (not including the points along the line) in the first quadrant defined by the lines $y = 3x$ and $y = \frac{1}{2}x$?

Show work to be graded below, and use the reverse side of the page to continue if necessary

PROBLEM 7

TEAM #

A rational number can be written $\frac{p}{q}$ where p and q are integers, $q \neq 0$, and p and q have no common factors. Let the function $f(x)$ be defined as follows: if x is irrational, then $f(x) = 0$. If x is rational, then $f(x) = \frac{1}{q}$. Prove that this function is continuous at every irrational.

Show work to be graded below, and use the reverse side of the page to continue if necessary