2007 ICMC Problems

1. Let p and q be distinct primes. Find a polynomial with integer coefficients that has $\sqrt{p}+\sqrt{q}$ as a root.

2. What is the value of the positive integer n for which the least common multiple of 36 and n is 500 greater than the greatest common divisor of 36 and n?

3. Evaluate: $\displaystyle\lim_{x\to\infty}(x+2)\cdot\int_x^{3x}\frac{dt}{t\sqrt{t^4+1}}$.

4. Answer the following.

   (a) Let p be a fixed prime. Suppose an integer a is selected at random. What is the probability that a is divisible by p? (Think about the possible remainders when dividing by p.)

   (b) Let p be a fixed prime. Suppose two integers a and b are selected at random. What is the probability that a and b are both divisible by p?

   (c) Suppose two integers a and b are selected at random. Show that the probability that a and b are relatively prime is $\displaystyle\prod_{p\in P}\left(1-\frac{1}{p^2}\right)$, where P is the set of all primes.

5. Let A be an $n\times n$ matrix such that $a_{ij}=1$ when $i\neq j$, and $a_{ij}=0$ when $i=j$. In other words, $A=\begin{bmatrix} 0 & 1 & 1 & \cdots & 1\\ 1 & 0 & 1 & \cdots & 1\\ 1 & 1 & 0 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \cdots & 0\end{bmatrix}$. Find A^{-1}. (Using the matrix $B=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & 1 & 1 & \cdots & 1\\ 1 & 1 & 1 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \cdots & 1\end{bmatrix}$ may be helpful.)

6. Let g and h be noncommuting elements in a group of odd order. If g and h satisfy the relations $g^3=e$ and $ghg^{-1}=h^3$, determine the order of h.