Indiana College Mathematics Competition
IPFW, April 1, 2005

1. Evaluate the integral

$$\int_{-1}^{1} \frac{|\sin(n\cos^{-1}x)|}{\sqrt{1-x^2}}dx \,.$$

2. All points in the plane are colored in red, white, or blue. Prove that there is at least one pair of points of the same color with distance between them one unit.

3. Find the limit of the sequence defined by

$$a_n = \frac{1}{n^3}\sum_{k=1}^{n} \ln(1+kn).$$

4. Let Q^+ be the set of all positive rational numbers and let "*" be an operation on Q^+ that satisfies the following conditions for all $a,b,c,d \in Q^+$:

$$(a*b)(c*d) = (ac)*(bd),$$
$$a*a = 1,$$
$$a*1 = a.$$

Compute the value of the expression $((6/5)*(8/15))*2$.

5. Consider the matrices A and B in $M_n(\Re)$ such that $A^3 = A^2$ and $A+B = I_n$. Show that the matrix $AB + I_n$ is nonsingular and find its inverse.

6. Determine the real constants *a*, *b*, *c*, and *p*, such that

$$\lim_{x\to\infty}\left[\sqrt{9x^4 - 24x^3 + 6x^2 + 5} - (ax^p + bx + c)\right] = \frac{7}{3}.$$

7. Let M and N be the midpoints of BC and CD in the parallelogram ABCD, and let P be the intersection of AM and BN. Determine the ratios $\frac{AP}{AM}$ and $\frac{BP}{BN}$.

8. Given the integers $x,\ y,$ and $z,$ prove that if 25 divides the sum $x^5 + y^5 + z^5,$ then 25 divides at least one of the numbers $x^5 + y^5,\ x^5 + z^5,$ or $y^5 + z^5$.